About $k$–Fibonacci Numbers and their Associated Numbers

Alvaro H. Salas

Universidad de Caldas, Manizales, Colombia
Universidad Nacional de Colombia
asalash2002@yahoo.com
FIZMAKO Research Group

Abstract

In this paper we define the associated $k$–Fibonacci numbers and we give a combinatorial interpretation for them.

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1 Introduction

Ever since Fibonacci modelled the growth of a rabbit population in his book Liber Abaci, the Fibonacci numbers and $\phi = \frac{1+\sqrt{5}}{2}$, a special constant related to them, have been applied countless times in art, science and architecture.

Besides the usual Fibonacci numbers many kinds of generalizations of these numbers have been presented in the literature(e.g. see [1-5] ). The well-known Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is defined as

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2$$

where $F_n$ denotes the $n$–th Fibonacci number.

For any positive real number $k$, the $k$–Fibonacci sequence, say $\{F_{n,k}\}_{n=0}^{\infty}$ is defined recurrently by

$$F_{n+1,k} = kF_{n,k} + F_{n-1,k}$$

with initial conditions

$$F_{0,k} = 0 \text{ and } F_{1,k} = 1.$$
In [2], these general $k-$Fibonacci numbers $\{F_{n,k}\}_{n=0}^{\infty}$ were found by studying the recursive application of two geometrical transformations used in the well known four-triangle longest-edge (4TLE) partition. Many properties of these numbers are obtained directly from elementary matrix algebra. In [3], many properties of these numbers are deduced and related with the so-called Pascal 2-triangle. In [4], authors defined $k-$Fibonacci hyperbolic functions similar to hyperbolic functions and Fibonacci hyperbolic functions. They deduced some properties of $k-$Fibonacci hyperbolic functions related with the analogous identities for the $k-$Fibonacci numbers. Finally, authors studied 3-dimensional $k-$Fibonacci spirals with a geometric point of view in [5]. Several properties of these new $k-$Fibonacci hyperbolic functions are studied in an easy way. In the 19th century, the French mathematician Binet devised two remarkable analytical formulas for the Fibonacci and Lucas numbers [6]. Some identities for the $k-$Fibonacci numbers may be found in [7].

Members of the $k-$Fibonacci sequence $\{F_{n,k}\}_{n=0}^{\infty}$ will be called $k-$Fibonacci numbers. Some of them are:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$F_{n,k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$k$</td>
</tr>
<tr>
<td>3</td>
<td>$k^2 + 1$</td>
</tr>
<tr>
<td>4</td>
<td>$k^3 + 2k$</td>
</tr>
<tr>
<td>5</td>
<td>$k^4 + 3k^2 + 1$</td>
</tr>
<tr>
<td>6</td>
<td>$k^5 + 4k^3 + 3k$</td>
</tr>
<tr>
<td>7</td>
<td>$k^6 + 5k^4 + 6k^2 + 1$</td>
</tr>
<tr>
<td>8</td>
<td>$k^7 + 6k^5 + 10k^3 + 4k$</td>
</tr>
<tr>
<td>9</td>
<td>$k^8 + 7k^6 + 15k^4 + 10k^2 + 1$</td>
</tr>
<tr>
<td>10</td>
<td>$k^9 + 8k^7 + 21k^5 + 20k^3 + 5k$</td>
</tr>
<tr>
<td>11</td>
<td>$k^{10} + 9k^8 + 28k^6 + 35k^4 + 15k^2 + 1$</td>
</tr>
<tr>
<td>12</td>
<td>$k^{11} + 10k^9 + 35k^7 + 56k^5 + 35k^3 + 6k$</td>
</tr>
<tr>
<td>13</td>
<td>$k^{12} + 11k^{10} + 45k^8 + 84k^6 + 70k^4 + 21k^2 + 1$</td>
</tr>
<tr>
<td>14</td>
<td>$k^{13} + 12k^{11} + 55k^9 + 120k^7 + 126k^5 + 56k^3 + 7k$</td>
</tr>
<tr>
<td>15</td>
<td>$k^{14} + 13k^{12} + 66k^{10} + 165k^8 + 210k^6 + 126k^4 + 28k^2 + 1$</td>
</tr>
<tr>
<td>16</td>
<td>$k^{15} + 14k^{13} + 78k^{11} + 220k^9 + 330k^7 + 252k^5 + 84k^3 + 8k$</td>
</tr>
<tr>
<td>17</td>
<td>$k^{16} + 15k^{14} + 91k^{12} + 286k^{10} + 495k^8 + 462k^6 + 210k^4 + 36k^2 + 1$</td>
</tr>
</tbody>
</table>

Particular cases of the previous definition are:

- If $k = 1$, the classic Fibonacci sequence is obtained:
  \[ F_n, F_1 = 0, 1, \text{ and } F_{n+1} = F_n + F_{n-1} \text{ for } n \geq 1; \]
  \[ \{F_n\}_{n=0}^{\infty} = \{0, 1, 1, 2, 3, 5, 8, \ldots\}. \]

- If $k = 2$, the classic Pell sequence appears:
  \[ P_0 = 0, P_1 = 1, \text{ and } P_{n+1} = 2P_n + P_{n-1} \text{ for } n \geq 1; \]
\{P_n\}_{n=0}^{\infty} = \{0, 1, 2, 5, 12, 29, 70, \ldots\}.

- If \(k = 3\), the following sequence appears:
  \[H_0 = 0, \ H_1 = 1 \text{ and } H_{n+1} = 3H_n + H_{n-1} \text{ for } n \geq 1;
  \{H_n\}_{n=0}^{\infty} = \{0, 1, 3, 10, 33, 109, \ldots\}.

2 The associated \(k\)--Fibonacci numbers and their combinatorial interpretation

We define the sequence \(\{A_{n,k}\}_{n=0}^{\infty}\) associated to \(\{F_{n,k}\}_{n=0}^{\infty}\) as

\[A_{0,k} = 1 \text{ and } A_{n,k} = F_{n,k} + F_{n-1,k} \text{ for } n = 1, 2, 3, \ldots\]

Observe that for \(n = 1, 2, 3, \ldots\) the expression \(A_{n,k}\) is the sum of the two consecutive \(k\)--Fibonacci numbers \(F_{n,k}\) and its predecessor \(F_{n-1,k}\). The members of the sequence \(\{A_{n,k}\}_{n=0}^{\infty}\) will be called associated \(k\)--Fibonacci numbers. An equivalent definition for the sequence \(\{A_{n,k}\}_{n=0}^{\infty}\) is

\[A_{n,k} = \begin{cases} 
1 & \text{if } n = 0 \\
1 & \text{if } n = 1 \\
(k+1)F_{n-1,k} + F_{n-2,k} & \text{if } n \geq 2 \end{cases}
\]

Observe that

\[A_{n,k} = F_{n,k} + F_{n-1,k} = kF_{n-1,k} + F_{n-2,k} + kF_{n-2,k} + F_{n-3,k} = k(F_{n-1,k} + F_{n-2,k}) + (F_{n-2,k} + F_{n-3,k}) = kA_{n-1,k} + A_{n-2,k}.
\]

This allows to define recursively the sequence of associated \(k\)--Fibonacci numbers as follows:

\[A_{n,k} = \begin{cases} 
1 & \text{if } n = 0 \\
1 & \text{if } n = 1 \\
kA_{n-1,k} + A_{n-2,k} & \text{if } n \geq 2 \end{cases}
\]
Following table shows some of the associated \( k \)-Fibonacci numbers

\[
\begin{array}{c|c}
 n & A_{n,k} \\
0 & 1 \\
1 & 1 \\
2 & k + 1 \\
3 & k^2 + k + 1 \\
4 & k^3 + k^2 + 2k + 1 \\
5 & k^4 + k^3 + 3k^2 + 2k + 1 \\
6 & k^5 + k^4 + 4k^3 + 3k^2 + 3k + 1 \\
7 & k^6 + k^5 + 5k^4 + 4k^3 + 6k^2 + 3k + 1 \\
8 & k^7 + k^6 + 6k^5 + 5k^4 + 10k^3 + 6k^2 + 4k + 1 \\
9 & k^8 + k^7 + 7k^6 + 6k^5 + 15k^4 + 10k^3 + 10k^2 + 4k + 1 \\
10 & k^9 + k^8 + 8k^7 + 7k^6 + 21k^5 + 15k^4 + 20k^3 + 10k^2 + 5k + 1 \\
11 & k^{10} + k^9 + 9k^8 + 8k^7 + 28k^6 + 21k^5 + 35k^4 + 20k^3 + 15k^2 + 5k + 1 \\
\end{array}
\]

For a fixed integer \( k \geq 1 \) we define

\[
f_{0,k} = 1 \quad \text{and} \quad f_{1,k} = 1 \quad \text{for} \quad k = 1, 2, 3, \ldots \quad (2)
\]

Let us consider the set \( S = \{0, \alpha_1, \alpha_2, \ldots, \alpha_k\} \) whose elements are 0 and \( k \) pairwise distinct symbols \( \alpha_1, \alpha_2, \ldots, \alpha_k \) such that \( \alpha_i \neq 0 \) \((i = 1, 2, \ldots, k)\).

For any \( n \geq 2 \) let \( f_{n,k} \) be the number of \((n-1)\)-permutations of the set \( S \) with repetition and the restriction that no two equal symbols \( \alpha_i \) are consecutive. We will call this restriction \( R \). We have the following

**Theorem.** For any \( k \geq 1 \) and any \( n \geq 2 \),

\[
f_{n,k} = kf_{n-1,k} + f_{n-2,k}.
\]

**Proof.** We will prove this by induction on \( n \). The theorem is valid for \( n = 2 \) since the number of 1-permutations of the set \( S = \{0, \alpha_1, \alpha_2, \ldots, \alpha_k\} \) with repetition satisfying restriction \( R \) equals \( k + 1 \). They are

\[
0, \alpha_1, \alpha_2, \ldots, \alpha_k
\]

Suppose that theorem holds for any \( n \) with \( 2 \leq n \leq m \). The \( m \)-permutations of the set \( S = \{0, \alpha_1, \alpha_2, \ldots, \alpha_k\} \) satisfying restriction \( R \) are divided into two mutually disjoint classes: the class \( I \) of the permutations ending with 00 and the class \( II \) of permutations for which the two last elements are distinct symbols of the set \( S \).

Given any permutation of the class \( I \) we obtain a \((m-2)\)-permutation satisfying restriction \( R \) if we delete the last two zeros in it. Conversely, if we have a \((m-2)\)-permutation satisfying restriction \( R \) we obtain a permutation of the class \( I \) by adding two zeros at the end of it. By the inductive hypothesis, the class \( I \) has \( f_{m-1,k} \) elements.
On the other hand, if $\beta_1\beta_2\cdots\beta_{m-1}$ is any $(m-1)$-permutation satisfying restriction $R$ we obtain $k$ permutations of class $II$ by adding at the end of it any element in the set $S_0 = \{0,\alpha_1,\alpha_2,\ldots,\alpha_k\} \setminus \{\beta_{m-1}\}$. The result is the $m$-permutation $\beta_1\beta_2\cdots\beta_{m-2}\beta_{m-1}\gamma$. Conversely, if we have a permutation $\beta_1\beta_2\cdots\beta_{m-2}\beta_{m-1}\beta_{m}$ of class $II$ we obtain a $(m-1)$-permutation satisfying restriction $R$ by deleting any element on it. In view of the inductive hypothesis, the class $II$ contains $kf_{m,k}$ elements. Thus

$$f_{m+1,k} = kf_{m,k} + f_{m-1,k}$$

Theorem has been proved.

From previous theorem and taking into account (1) and (2) we conclude that

$$A_{n,k} = f_{n,k} \text{ for all } n \geq 1 \text{ and } k \geq 1.$$ 

We obtained an interpretation of the associated $k$–Fibonacci numbers in terms of permutations. Since a permutation generates a word we may interpret these numbers in terms of words of given length. As an illustration, Table 1 shows some of these permutations for $k = 1$. Observe that $f_{n,1}$ is the number of $(n-1)$–permutations of the set $S = \{0,1\}$ with repetition and the restriction that no two 1 are consecutive ($n = 2, 3, 4, 5,\ldots$).

<table>
<thead>
<tr>
<th></th>
<th>$f_{2,1}$</th>
<th>$f_{3,1}$</th>
<th>$f_{4,1}$</th>
<th>$f_{5,1}$</th>
<th>$f_{6,1}$</th>
<th>$f_{7,1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
</tr>
<tr>
<td>00 01 10</td>
<td>00 001 011 100 101</td>
<td>000 010 010 100 101 0101</td>
<td>000 001 0010 00101 01000 01001 01010 10001 10010 10101</td>
<td>00000 00001 00010 000100 000101 001000 001001 001010 010000 010001 010010 010100 010101 100000 100010 100100 100101 101000 101001 101010 101010 101011</td>
<td>000000 000001 000010 000100 000101 001000 001001 001010 010000 010001 010010 010100 010101 100000 100010 100100 100101 101000 101001 101010 101010 101011</td>
<td></td>
</tr>
</tbody>
</table>

Table 1. Number of permutations $f_{n,k}$ for $k = 1$ and $n = 2, 3, 4, 5, 6, 7$. 
From Table 1 we see that

\[ f_{n,1} = f_{n-1,1} + f_{n-2,1} \quad (n = 2, 3, 4, 5, 6, 7) \]

where

\[ f_{0,1} = 1 \text{ and } f_{1,1} = 1. \]

In other words, \( f_{n,1} \) are the well known Fibonacci numbers. They are in essence the same associated Fibonacci numbers.

On the other hand, on Table 2 we may observe some permutations associated to the numbers \( f_{n,k} \) for \( k = 2 \). Observe that \( f_{n,2} \) is the number of \((n - 1)\)–permutations of the set \( S = \{0, 1, 2\} \) with repetition and the restriction that no two 1 and no two 2 are consecutive \((n = 2, 3, 4, 5, \ldots)\).

\[
\begin{align*}
&f_{2,2} = 3 = 2f_{1,2} + f_{0,2} \\
&f_{3,2} = 7 = 2f_{2,2} + f_{1,2} \\
&f_{4,2} = 17 = 2f_{3,2} + f_{2,2} \\
&f_{5,2} = 41 = 2f_{4,2} + f_{3,2} \\
&f_{6,2} = 99 = 2f_{5,2} + f_{4,2}
\end{align*}
\]
Table 2. Number of permutations \( f_{n,k} \) for \( k = 2 \) and \( n = 2, 3, 4, 5, 6 \).

From Table 2 we see that

\[
f_{n,2} = 2f_{n-1,2} + f_{n-2,2} \quad (n = 2, 3, 4, 5, 6)
\]

where

\[
f_{0,2} = 1 \text{ and } f_{1,2} = 1.
\]

In other words, \( f_{n,2} \) are the associated Pell numbers.

3 References


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Given two numbers N and K, find the number of ways to represent N as the sum of K Fibonacci numbers. Examples: Input : n = 12, k = 1 Output : 0. Input : n = 13, k = 3 Output : 2 Explanation : 2 + 3 + 8, 3 + 5 + 5. Recommended: Please try your approach on IDE first, before moving on to the solution. Approach: The Fibonacci series is f(0)=1, f(1)=2 and f(i)=f(i-1)+f(i-2) for i>1. Let F(x, k, n) be the number of ways to form the sum x using exactly k numbers from f(0), f(1), ..., f(n-1). To find a recurrence for F(x, k, n), notice that there are two cases: whether f(n-1) in the sum or ... If x<0 because there's no way to form a negative sum using a finite number of nonnegative numbers. Below is the implementation of above approach Fibonacci numbers are ubiquitous in nature. They are everywhere from the spiraling arms of a galaxy to the energy levels within an atom. Image by Karin Henseler from Pixabay. Fibonacci Numbers and Generating Functions. How to use a power series to find the general term for the celebrated sequence. Adam Hrankowski. By why limit yourself to integers or even real numbers as input? What is the i-th Fibonacci number? What does that even mean? Perhaps such questions are fodder for another article. Fibonacci Numbers and Nature. This page has been split into TWO PARTS. This, the first, looks at the Fibonacci numbers and why they appear in various "family trees" and patterns of spirals of leaves and seeds. Here we follow the convention of Family Trees that parents appear above their children, so the latest generations are at the bottom and the higher up we go, the older people are. Such trees show all the ancestors (predecessors, forebears, antecedents) of the person at the bottom of the diagram. One plant in particular shows the Fibonacci numbers in the number of "growing points" that it has. Suppose that when a plant puts out a new shoot, that shoot has to grow two months before it is strong enough to support branching. Every Fibonacci number bigger than 1 [except F(6)=8 and F(12)=144] has at least one prime factor that is not a factor of any earlier Fibonacci number. Those factors are shown like this. Index numbers that are prime are shown like this. The first 300 Fibonacci numbers n : F(n)=factorisation 0 : 0 1 : 1 2 : 1 3 : 2 4 : 3 5 : 5 6 : 8 = 23 7 : 13 8 : 21 = 3 x 7 9 : 34 = 2 x 17 10 : 55 = 5 x 11 11 : 89 12 : 144 = 24 x 32 13 : 233 14 : 377 = 13 x 29 15 : 610 = 2 x 5 x... It calculates thousands of Fibonacci numbers exactly and millions upon millions to the first few digits! The Puzzling World of the Fibonacci Numbers. the Fibonacci Home Page. Associated k-Fibonacci numbers. 2477. On the other hand, if I1I2A···An is any (m−1)-permutation satisfying restriction R we obtain k permutations of class II by adding at the end of it any element in the set S0 = {0, l±1, l±2, ..., l±k} (IÎ“−1). Associated k-Fibonacci Numbers and Their Applications. Applications of Fibonacci numbers proceedings of the Third. On Fibonacci numbers and their applications.