1. Introduction

2. What can we assume for students’ background?

We assume the students know more about algebra than geometry. We assume basic skills of algebra up to the achievement of a standard *College Algebra* course, *i.e.*, we assume the students know how to solve linear and quadratic equations in one unknown, the latter by easy factorization or by using the quadratic formula. But we do not assume the students comfortable with the process of *completing the squares*. We assume the students have at ease with equations with numerical coefficients, but usually at a loss when the coefficients involving letters.

As for geometry, we assume the students know the Pythagorean theorem, but usually not its proof. We assume the most basic facts of euclidean geometry, say, that the angle sum of a triangle is $180^\circ$, but not the fact that an external angle of a triangle is the sum of the two remote internal angles. We assume the students know the concept of congruence of triangles and the several tests for congruence. We assume they know the notion of similarity, but with some difficulty in building the equalities of proportions of sides that can be read from a pair of similar triangles. We assume they have some acquaintance with the notion of perpendicular bisectors, angle bisectors, circumcenter, centroid, and the like, but do not know how they relate.

As for trigonometry, we assume the students know the trigonometric functions as circular functions defined through a unit circle in the Cartesian plane, their periodicity and some simple identities including the addition formulas and the double angle formulas. But we do not assume that they can do trigonometry with right-angled triangles or more general triangles.

For geometric constructions, we do not assume any experience nor any prior exposure. The students certainly know that one can draw straight lines
by joining two points with a straightedge and a circle given its center and one point on the circumference.

3. **What can we hope to achieve?**

4. **The first 6 books of Euclid’s *Elements* through an eye of geometric constructions**

We shall unashamedly teach Euclid’s *Elements* as a book on *how to construct certain geometric figures efficiently and accurately using ruler and compass, and ascertaining the validity*. The first three postulates before Book 1 are on the basic use of the ruler and the compass.

- **Postulate 1**: To draw a straight line from any point to any point.
- **Postulate 2**: To produce a finite straight line continuously in a straight line.
- **Postulate 3**: To describe a circle with any center and radius.

Here are the propositions of the first 6 books of Euclid’s *Elements* pertaining to constructions.

<table>
<thead>
<tr>
<th>Book</th>
<th>Propositions</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>1,2,3,9,10,11,12,22,23,31,42,44,45,46.</td>
</tr>
<tr>
<td>III</td>
<td>1,17,30.</td>
</tr>
<tr>
<td>IV</td>
<td>entire book of 16 propositions.</td>
</tr>
<tr>
<td>VI</td>
<td>9,10,11,12,13,25,28,29,30.</td>
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</table>

The interpersing propositions can be regarded as supplementing and supporting the validity of these constructions. I divide these constructions in the *Elements* into two classes. The first class consists of those fundamental constructions which the students should master. These include

4.1. **Basic constructions.**

- **I.1**: To construct an equilateral triangle on a given finite straight line.
- **I.9**: To bisect a given rectilinear angle.
- **I.10**: To bisect a given finite straight line.
- **I.11**: To draw a straight line at right angles to a given straight line from a given point on it.
- **I.12**: To draw a straight line perpendicular to a given infinite straight line from a given point not on it.
- **I.23**: To construct a rectilinear angle equal to a given rectilinear angle on a given straight line and at a point on it.
- **I.31**: To draw a straight line through a given point parallel to a given straight line.
I.42: To construct a parallelogram equal to a given triangle in a given rectilinear angle.
I.46: To describe a square on a given straight line.
I.47*: In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.
III.1: To find the center of a given circle.
III.17: From a given point to draw a straight line touching a given circle.
VI.10: To cut a given uncut straight line similarly to a given cut straight line.
VI.11: To find a third proportional to two given straight lines.
VI.12: To find a fourth proportional to three given straight lines.
VI.13: To find a mean proportional to two given straight lines.
VI.30: To cut a given finite straight line in extreme and mean ratio.

Some of these constructions can be done very easily with the built-in commands of the Geometer’s Sketchpad®. But it is instructive to explain the basic principles. In doing this, the students are exposed to the use of following two basic principles in validating geometrical assertions:
(1) the external angle theorem of a triangle, and
(2) tests of congruence of triangles (SAS, SSS, ASA, AAS, RHS).

4.2. Constructions in Euclid’s Elements as exercises. There are a number of the constructions in Euclid’s Elements suitable as exercises for the students experimenting with the use of the Geometer’s Sketchpad®.

IV.6: To inscribe a square in a given circle.
IV.7: To circumscribe a square about a given circle.
IV.8: To inscribe a circle in a given square.
IV.9: To circumscribe a circle about a given square.

The students should have no problem in performing these constructions using the Geometer’s Sketchpad®. They have, however, to learn how to describe the constructions and justify them. In this process, they learn that it is too cumbersome to resort to the basic congruence test every time. They’d better know some of the basic properties of simple geometric figures (that can be justified by these congruence test) and use them freely. For example, the following are among the basic properties of a parallelogram.

4.3. Discussion on some special propositions. Discussion of some seemingly naive constructions in the Elements. Why did Euclid want to perform the following?

I.2: To place a straight line equal to a given straight line with one end at a given point.
I.3: To cut off from the greater of two given unequal straight lines a straight line equal to the less.

The use of I.47 in the construction of geometric means.

4.4. Solution of quadratic equations by geometric constructions.

4.4.1. The intersecting chords theorem.

4.4.2. The use of the quadratic formula.

4.5. Basic constructions not in Euclid’s Elements.

4.5.1. The harmonic mean.

5. Metrical theorems

5.1. The sine and cosine formulas.

5.2. Stewart’s theorem.

5.3. The power of a point with respect to a circle.

5.4. Menelaus and Ceva theorems.

6. Problem solving in plane geometry

Although plane geometry has been excluded from much of high school and undergraduate curricula, it is surprising that interests in such problems never faded. Here is a simple count of the geometry problems in various problem journals since 1975:

AMM MG CMJ PME CM

I choose a number of problems (about 100) from these journals. In each case, the students can make a simple diagram using the Geometer's Sketchpad®. Experimenting with the dynamic software will help them understand the problems better and suggest a solution to the problem. Some of these problems may suggest other problems or constructions. For 50 of these problems I provide a diagram.
7. Some famous problems

- Find the length of a side of an equilateral triangle in which the distances from its vertices to an interior point are 5, 7, 8.

Long solution by H. Eves. See also CM10.p242, Bottema, On the distances of a point to the vertices of a triangle. Here is a generalization:

For three positive numbers \(a, b, c\) satisfying \(a \leq b \leq c \leq a + b\), give a euclidean construction of an equilateral triangle \(ABC\) together with a point \(P\) (not necessarily inside the triangle) such that \(AP = a, BP = b\) and \(CP = c\). Distinguish between the cases \(a + b = c\) and \(a + b < c\).

Solution. (a) If \(a + b = c\), let \(O, A, C, B\) be points on the plane such that \(OA = a, OC = a + b = c, OB = b, \angle AOC = \angle COB = 60^\circ\) and \(\angle AOB = 120^\circ\). Then \(ABC\) is an equilateral triangle.

(b) Suppose \(a + b < c\). Let \(XYZ\) be a triangle with \(YZ = \sqrt{3}a, ZX = \sqrt{3}b\) and \(XY = \sqrt{3}c\). Let \(A, B\) and \(C\) be the centers of the equilateral triangles with sides \(YZ, ZX\) and \(XY\) respectively drawn outside the triangle. It is well known that \(ABC\) is an equilateral triangle. The circumcircle of the equilateral triangles with centers \(A, B, C\) intersect at a point \(P\). Clearly, \(AP = AX = a\), and similarly, \(BP = b, CP = c\). A second equilateral triangle \(A'B'C'\) and a second point \(P'\) is obtained by constructing the equilateral triangles “inside” \(XYZ\).

8. Some simple problems for exploration

- Animate a point \(P\) on the base \(BC\) of an isosceles triangle \(ABC\). Let \(Y\) and \(Z\) be the pedals of \(P\) on \(AC\) and \(AB\) respectively. What can you say about the sum of the lengths \(PY\) and \(PZ\)? Give an explanation.

9. Elegant geometric constructions

- Mixtilinear incircles
- Archimedes’ twin circles of the shoemaker’s knife
- Apollonian problem


- To construct a trapezoid; given the bases, the perpendicular distance between the bases and the angle formed by the diagonals.
9.2. Two projects of constructions of triangles. (1) Triangles with three given points
   (2) Triangles with three given lengths

10. Relations with number theory

10.1. Right triangles with perimeter and area equal. See also CMJ232.825, where it is asserted that there are only five triangles of integer sides, with area equal to perimeter. Two of these are right triangles, namely (6,8,10) and (5,12,13). This note shows that these are the only ones that contain right angles.

   Conjecture: For every natural number $n$, there is at least one primitive Pythagorean triangle in which the area equals $n$ times the perimeter. See MG1088.795.S811, MG1077.794.S804, where this conjecture was resolved.

   CMJ354.873.S893. Let $ABC$ be a right triangle. Let $CP$ and $CQ$ be, respectively, the median and altitude of the hypotenuse $AB$. Under what condition are the sides of $\triangle CPQ$ integers?

   CMJ390.885.902. Find the smallest Pythagorean triangle in which a square with integer sides can be inscribed so that a side of the square coincides with the hypotenuse of the triangle.

   In what sense? See also CMJ64.762.S774, MG945.754.S764.

11. Triangle geometry

   CMJ408.894.408.S905. Let $A'$, $B'$, $C'$ be the excenters of triangle $ABC$. The perpendiculars from $B'$ to $AB$ and from $C'$ to $AC$ meet in a point $A''$; points $B''$ and $C''$ are determined analogously. Prove that the lines $AA''$, $BB''$, $CC''$ are concurrent.

   CMJ417.901.S911. Let $ABC$ be a triangle, the lengths of whose sides are $a, b, c$. Let $I$ denote the incenter and $E$ the excenter opposite the vertex $A$. Let $P$ and $Q$ ($S$ and $R$) be points on the sides (extensions of the sides) $AB$ and $AC$ such that segment $PQ$ ($SR$) is parallel to side $BC$.

   (a) Prove that (i) $PQ = PB + QC$ if and only if $PQ$ passes through $I$, and (ii) $SR = SB + RC$ if and only if $SR$ passes through $E$.

   (b) Determine the lengths of the sides of the trapezoid $PQRS$ in terms of $a, b$ and $c$. 
Characterize those triangles such that the angle bisectors from one vertex, the median from a second vertex, and the altitude from the third vertex are concurrent.

Solution. \( \cos A = \frac{c}{2s} \).

See also AMME263.37p104,599 concerning the euclidean construction of this triangle.

(CMJ558.954.S964. (J.Fukuta).) Prove that for any triangle \( ABC \), there exists one and only one set of points \( D, E, F \) satisfying:

(a) \( D \) lies on side \( BC \), \( E \) lies on side \( CA \), and \( F \) lies on side \( AB \);

(b) \( EA + AF = BC; FB + BD = CA; DC + CE = AB \); and

(c) \( AD, BE, \) and \( CF \) are concurrent.

Solution. Denote by \( a, b, c \) the lengths of the sides of the triangle: \( BC = a, CA = b, \) and \( AB = c \), and by \( s = \frac{1}{2}(a + b + c) \) the semi-perimeter.

First note that if \( D, E, \) and \( F \) are respectively the points of contact of the sides \( BC, CA, \) and \( AB \) with the excircles of the triangle on the opposite sides of \( A, B, \) and \( C, \) it is easy to establish

\[
EA = BD = s - c; \quad FB = CE = s - a; \quad DC = AF = s - b.
\]

Condition (b) is clearly satisfied. Also, \( AD, BE, \) and \( CF \) are concurrent by Ceva’s theorem.

\[
\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{s - b}{s - a} \cdot \frac{s - c}{s - b} \cdot \frac{s - a}{s - c} = 1.
\]

The intersection of \( AD, BE \) and \( CF \) is usually called the Nagel point of the triangle.

Now, we show that this is the only set of points satisfying the conditions (a), (b), (c). Let \( D', E', \) and \( F' \) be a set of points satisfying the same conditions. Suppose \( \overline{AF'} = \overline{AF} + \epsilon \) for some \( \epsilon \). Then

\[
\overline{E'A} = s - c - \epsilon, \quad \overline{BD'} = s - c + \epsilon; \quad \overline{F'B} = s - a - \epsilon, \quad \overline{CE'} = s - a + \epsilon; \quad \overline{D'C} = s - b - \epsilon, \quad \overline{AF'} = s - b + \epsilon.
\]

If \( AD', BE', \) and \( CF' \) are to be concurrent, then Ceva’s theorem requires

\[
\frac{AF'}{F'B} \cdot \frac{BD'}{D'C} \cdot \frac{CE'}{E'A} = 1.
\]

This means

\[
(s - a + \epsilon)(s - b + \epsilon)(s - c + \epsilon) = (s - a - \epsilon)(s - b - \epsilon)(s - c - \epsilon),
\]

\[
\epsilon[\epsilon^2 + (s - a)(s - b) + (s - b)(s - c) + (s - c)(s - a)] = 0.
\]
Since \((s - a)(s - b) + (s - b)(s - c) + (s - c)(s - a) > 0\). This requires \(\epsilon = 0\). This means the points \(D', E', \) and \(F'\) coincide respectively with the points \(D, E, \) and \(F\).

CMJ595.971.S981. (J.B.Romero Márquez). In a right triangle whose sides are \(a, b,\) and \(c\) (with \(a \leq b < c\)), evaluate

\[
\lim_{b \to a} \frac{m_b - m_a}{\theta_b - \theta_a},
\]

where \(m_a, m_b, \theta_a\) and \(\theta_b\) are the lengths of the medians and the angle bisectors meeting sides \(a\) and \(b\) respectively.

**Solution.** We solve a slightly more generally problem. Let \(a\) and \(c\) be fixed, and \(r := \frac{c}{a}\). The medians \(m_a\) and \(m_b\) are given by Apollonius’ Theorem:

\[
m_a^2 = \frac{1}{4}(2b^2 + 2c^2 - a^2) \quad \text{and} \quad m_b^2 = \frac{1}{4}(2c^2 + 2a^2 - b^2).
\]

It follows that

\[
\lim_{b \to a} \frac{m_b^2 - m_a^2}{a - b} = \lim_{b \to a} \frac{3}{4}(a + b) = \frac{3}{2}a.
\]

Since \(\lim_{b \to a}(m_b + m_a) = \lim_{b \to a}\sqrt{2c^2 + 2a^2 - b^2} = \sqrt{2c^2 + a^2} = a\sqrt{2r^2 + 1},\)

we have

\[
\lim_{b \to a} \frac{m_b - m_a}{a - b} = \frac{3a}{2\sqrt{2c^2 + a^2}} = \frac{3}{2\sqrt{2r^2 + 1}} \quad (1)
\]

The angle bisectors \(\theta_a\) and \(\theta_b\) are given by

\[
\theta_a^2 = bc \left(1 - \left(\frac{a}{b + c}\right)^2\right) \quad \text{and} \quad \theta_b^2 = ca \left(1 - \left(\frac{b}{c + a}\right)^2\right).
\]

Here,

\[
\frac{\theta_b^2 - \theta_a^2}{a - b} = \frac{c(a + b + c)(a^2b + ab^2 + 3abc + ac^2 + be^2 + c^3)}{(a + c)^2(b + c)^2},
\]

\[
\lim_{b \to a} \frac{\theta_b^2 - \theta_a^2}{a - b} = \frac{c(c + 2a)(c^3 + 2ac^2 + 3a^2c + 2a^3)}{(c + a)^4}
\]

\[
= \frac{c(c + 2a)(c^2 + ac + 2a^2)}{(c + a)^3}
\]

\[
= \frac{c(c + 2a)(r^2 + r + 2)}{(r + 1)^3}.
\]

Since

\[
\lim_{b \to a}(\theta_b + \theta_a) = 2c \sqrt{a \left(1 - \left(\frac{a}{a + c}\right)^2\right)} = 2a \sqrt{r \left(1 - \left(\frac{1}{r + 1}\right)^2\right)} = \frac{2ar}{r + 1} \sqrt{r + 2},
\]
we have
\[ \lim_{b \to a} \frac{\theta_b - \theta_a}{a - b} = \frac{r^2 + r + 2}{2(r + 1)^2} \sqrt{r + 2}. \tag{2} \]

Combining (1) and (2), we have
\[ \lim_{b \to a} \frac{m_b - m_a}{\theta_b - \theta_a} = \frac{3(r + 1)^2}{r^2 + r + 2 \sqrt{r + 2}(2r^2 + 1)}. \]

For a right triangle \( r = \frac{c}{a} = \sqrt{2} \), we have
\[ \lim_{b \to a} \frac{m_b - m_a}{a - b} = \frac{3}{2\sqrt{5}} \approx 0.67082 \ldots, \]
\[ \lim_{b \to a} \frac{\theta_b - \theta_a}{a - b} = \sqrt{17 - \frac{23}{\sqrt{2}}} \approx 0.858221 \ldots \]
\[ \lim_{b \to a} \frac{m_b - m_a}{\theta_b - \theta_a} = \frac{3}{14} \sqrt{\frac{1}{5}(34 + 23\sqrt{2})} \approx 0.78164 \ldots. \]

(1) In a circle there are three parallel chords \( A_1A_2, B_1B_2, C_1C_2 \). Show that the orthocenters of the eight triangles \( A_iB_jC_k, i, j, k = 1, 2 \), are collinear.

(2) Let \( I \) be the incenter of triangle \( ABC \), \( r_1 \) the inradius of triangle \( IAB \) and \( r_2 \) the inradius of triangle \( IAC \). Computer experiments using Geometer’s Sketchpad suggest that \( r_2 < \frac{5}{4}r_1 \).

Prove or disprove the conjecture.

Can \( \frac{5}{4} \) be replaced by a smaller constant?

**Solution.** We prove the conjecture with the constant \( \frac{5}{4} \) improved to \( \frac{\sqrt{2} + 1}{2} \).

Given triangle \( ABC \) with incenter \( I \) and inradius \( r \), the perimeter of triangle \( IBC \) is
\[ r \left( \cot \frac{B}{2} + \csc \frac{B}{2} + \cot \frac{C}{2} + \csc \frac{C}{2} \right) = r \left( \cot \frac{B}{4} + \cot \frac{C}{4} \right) = \frac{r \sin \left( \frac{B}{4} + \frac{C}{4} \right)}{\sin \frac{B}{4} \sin \frac{C}{4}}. \]

The inradius of the same triangle is
\[ r_a = \frac{\frac{1}{2}ra}{\frac{1}{2}r \left( \cot \frac{B}{4} + \cot \frac{C}{4} \right)} = \frac{a \sin \frac{A}{4} \sin \frac{B}{4} \sin \frac{C}{4}}{\sin \frac{A}{4} \sin \left( \frac{B}{4} + \frac{C}{4} \right)} \]
\[ = \frac{2a \sin \frac{A}{4} \sin \frac{B}{4} \sin \frac{C}{4}}{\cos \left( \frac{B}{4} + \frac{C}{4} - \frac{A}{4} \right) - \cos \left( \frac{B}{4} + \frac{C}{4} + \frac{A}{4} \right)} = \frac{2a \sin \frac{A}{4} \sin \frac{B}{4} \sin \frac{C}{4}}{\cos \left( \frac{A}{4} - \frac{B}{4} \right) - \cos \frac{A}{4}.} \]
With \( f(x) = \frac{\sin x}{\cos \left( \frac{x}{4} - \frac{x}{2} \right) - \cos \frac{x}{4}} \), we write this inradius and those of triangles \( ICA, IAB \) in the form

\[
\begin{align*}
  r_a &= k \cdot f(A), \\
  r_b &= k \cdot f(B), \\
  r_c &= k \cdot f(C),
\end{align*}
\]

where \( k = 4R \sin \frac{A}{4} \sin \frac{B}{4} \sin \frac{C}{4}, \) \( R \) being the circumradius of triangle \( ABC \). It is easy to see that for \( 0 < x < \pi \),

(a) \( f(x) = f(\pi - x) \) for \( 0 < x < \pi \),

(b) \( f(x) \) is increasing for \( 0 < x \leq \frac{\pi}{2} \),

(c) \( \lim_{x \to 0} f(x) = 2 \sqrt{2} \),

(d) \( f\left(\frac{\pi}{2}\right) = 2 + \sqrt{2} \).

From these, we conclude that \( \frac{r_a}{r_c} = \frac{f(B)}{f(C)} < \frac{2+\sqrt{2}}{2\sqrt{2}} = \frac{\sqrt{2}+1}{2} < \frac{5}{4} \).

E396.398.S407. (D.L.MacKay). Given a triangle \( ABC \), construct a point \( X \) such that the three lines drawn through \( X \), each parallel to a side of the triangle and limited by the other two sides, are equal.

E411.403.S4010. (J.H.Butchart). Prove that, if the sides of a triangle form an arithmetic progression, the line joining the centroid to the incenter is parallel to one side.

Almost trivial proof using barycentric coordinates.

Design an animation picture illustrating this theorem.

Consider a triangle \( ABC \) with fixed base \( BC \). How can one construct \( A \) so that \( AB + AC = 2BC? \) We can draw a line through \( BP \) with length \( 2BC \). On this line we want to take a point \( A \) such that \( AP = AC \). This means that \( P \) lies on the perpendicular bisector of \( CP \). Clearly, \( P \) lies on the circle, center \( B \), radius \( 2BC \). Thus we obtain the locus of \( A \).

E467.414.S421.(V.Thébault). In a given triangle, show that the radical axes of the circumcircle with the respective circles whose diameters are the three medians meet the corresponding sides in three collinear points.

E1080.537.S543. (V.Thébault). Let \( I \) be the incenter, \( N \) thenine-point center, and \( D \) the midpoint of side \( BC \) of \( \triangle ABC \). Show that one of the common tangents to the circles \( I(N) \) and \( D(N) \) is parallel to \( BC \).

12. Constructions to identify triangle centers

The students are introduced to a number of basic triangle centers.

(1) Let \( X_a \) be the midpoint of the \( A \)-altitude, and \( Y_a \) the point of tangency of the \( A \)-excircle with \( BC \). Similarly define \( X_b, X_c, Y_b, Y_c \). The three lines \( X_aY_a, X_bY_b, X_cY_c \) intersect at a point. What is this intersection? (incenter)

(2) What if \( Y_a, Y_b, Y_c \) are the points of tangency of the incircle with the sides? \( (X_{57}) \)
13. Design of animation pictures

(1) Given a segment $BC$, construct a triangle $ABC$ whose Nagel point lies on the incircle.

The condition is that $b + c = 3a$.

References

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Euclidean geometry is one of the oldest branches of mathematics—the properties of different shapes have been investigated for thousands of years. For this reason, many tend to believe that today it is almost impossible to discover new properties and new directions for research in Euclidean geometry. In the present paper, we define the concepts of Pascal points, a circle that forms Pascal points, and a circle coordinated with the Pascal points formed by it, and we shall prove nine theorems that describe the properties of Pascal points on the sides of a convex quadrilateral. Projecting a sphere to a plane.

Outline. History. Geometers. v. t. e. Euclidean geometry is a mathematical system attributed to Alexandrian Greek mathematician Euclid, which he described in his textbook on geometry: the Elements. Euclid's method consists in assuming a small set of intuitively appealing axioms, and deducing many other propositions (theorems) from these. Although many of Euclid's results had been stated by earlier mathematicians, Euclid was the first to show how these propositions could be discovered after Euclid's death but is still built on Euclid's work. It is to be distinguished from non-Euclidean geometry, which is geometry based on axioms that are different from those used by Euclid. Throughout the centuries since Euclid lived, geometers have continued to develop Euclidean geometry and have discovered large numbers of interesting relationships. Since this model is built within Euclidean geometry, it is an appropriate topic for study in a course on Euclidean geometry. Euclidean constructions, mostly utilizing inversions in circles, are used to illustrate many of the standard results of hyperbolic geometry.