INVARIENTS OF ALGEBRAIC GROUPS

ALEXANDER MERKURJEV

For a central simple algebra $A$ of dimension 16 over a field $F$ (char $F \neq 2$) and exponent 2 in the Brauer group of $F$ (hence $A$ is a biquaternion algebra, i.e. the tensor product of two quaternion algebras by an old theorem of Albert [1, p.369]), M. Rost has constructed an exact sequence (cf. [13])

$$0 \longrightarrow SK_1(A) \xrightarrow{r} H^4(F, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^4(F(X), \mathbb{Z}/2\mathbb{Z}),$$

where $X$ is the Albert quadric of $A$. This result has been used in [14] to show that there is a field extension $E/F$ such that $SK_1(A \otimes_F E) \neq 0$ provided $A$ is a division algebra and deduce that the variety of the algebraic group $SL_1(A)$ is not rational. Since $SK_1(A) = SL_1(A)/[A^*, A^*]$, the map $r$ can be viewed as a natural (with respect to field extensions) collection of homomorphisms from the group of points of the algebraic group $SL_1(A)$ over field extensions of $F$ to the fourth cohomology group with coefficients $\mathbb{Z}/2\mathbb{Z}$. The result quoted above shows that if $A$ is division, then $r$ is a nontrivial collection of homomorphisms, being considered for all field extensions (the group $SK_1(A)$ can be trivial over the base field $F$ even if $A$ is a division algebra).

The aim of the present paper is to study natural with respect to field extensions group homomorphisms of a given algebraic group to cohomology-like groups. More precisely, for an algebraic group $G$ over a field $F$ and a cycle module $M$ (cf. [24]; for example, $M$ can be given by Galois cohomology groups) we introduce a notion of invariant of $G$ in $M$ of dimension $d$ as a natural transformation of functors $G \rightarrow M_d$ from the category $F$-fields to Groups. For example, the Rost’s map $r$ can be considered as an invariant of $SL_1(A)$ of dimension 4 in the cycle module $H^*[\mathbb{Z}/2\mathbb{Z}]$.

All invariants of $G$ in $M$ of degree $d$ form an abelian group, which we denote $Inv^d(G, M)$. We prove (Theorem 2.3) that if a cycle module $M$ is of bounded exponent, then the group $Inv^d(G, M)$ is isomorphic to the subgroup of multiplicative elements in the unramified group $A^0(G, M_d)$.

In section 3 we consider cohomological invariants, i.e. invariants in the cycle module $H^*[N]$ where $N$ is a Galois module over $F$ and compute the groups of invariants of dimensions 0, 1 for any $N$ and dimension 2 for $N = \mu_n^{\otimes -1}$ (Theorems 3.1, 3.4 and 3.13). It turns out that for a simply connected groups all these invariants are trivial. In section 4 we show that the group $Inv^3(G, H^*[\mu_n^{\otimes -1}])$ is also trivial for a simply connected $G$ (Proposition 4.9). Thus, the Rost’s

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invariant $r$ is an example of a nontrivial cohomological invariant of $\text{SL}_1(A)$ of the smallest dimension!

In section 5 we show that the group $\text{SL}_1(A)$ has no nontrivial invariants if $\text{ind}(A) \leq 2$. If $\text{ind}(A) = 4$ and $\exp(A) = 2$ (i.e. $A$ is a biquaternion division algebra), the Rost’s invariant appears to be the only nontrivial invariant of $\text{SL}_1(A)$ (Theorem 5.4).

In section 6 we generalize Rost’s theorem to the case of arbitrary central simple algebra $A$ of dimension 16 (without any restriction on the exponent). It turns out that the Rost’s exact sequence does not exist if $A$ is of exponent 4. Nevertheless, there exists an exact sequence (Theorem 6.6)

$$0 \to \text{SK}_1(A) \xrightarrow{\gamma} H^4(F, \mathbb{Z}/2\mathbb{Z})/(2[A] \cup H^2(F, \mathbb{Z}/2\mathbb{Z})) \to H^4(F(X), \mathbb{Z}/2\mathbb{Z}),$$

where $X$ is the generalized Severi-Brauer variety $SB(2, A)$. The map $r$ in this sequence can be considered as an invariant of $\text{SL}_1(A)$ in the cycle module

$$M_d(E) = H^d(E, \mathbb{Z}/2\mathbb{Z})/(2[A_E] \cup H^{d-2}(E, \mathbb{Z}/2\mathbb{Z})),
$$

which is not a cohomological cycle module. This statement motivates the definition of invariants in arbitrary cycle modules (not only cohomological ones).

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1. Notation and preliminary results

1.1. Algebraic groups. An algebraic group over a field $F$ is a smooth affine group scheme of finite type over $F$. The category of all algebraic groups over $F$ is denoted $F$-groups.

We write $F$-alg for the category of commutative $F$-algebras and Groups for the category of groups. We consider an algebraic group $G$ as a functor $G : F$-alg $\to$ Groups, taking a commutative $F$-algebra $A$ to the group of $A$-points $G(A) = \text{Mor}(\text{Spec}(A), G)$. If $A$ is an $F$-subalgebra of $B$, we identify $G(A)$ with a subgroup of $G(B)$.

Denote $F[G]$ the $F$-algebra of regular functions on $G$ and $F(G)$ the field of rational functions if $G$ is connected.

$G^* = \text{Hom}(G, \mathbb{G}_m)$ is the character group of $G$ over $F$.

Any point $g$ of the scheme $G$ defines an element of the group $G(F(g))$, which we also denote $g$. If $G$ is connected, the generic element of $G$ defines an element $\xi$ of $G(F[G])$. We also write $\xi$ for its image in $G(F(G))$.

$F$-fields is the full subcategory of $F$-alg consisting of fields.

$\text{Ab}$ is the category of abelian groups.

Let $G$ be an algebraic group over a field $F$. Denote $O$ the power series ring $F[[t]]$ and $O_m$ the factor ring $O/t^mO$. For any element $g$ of $G(O)$ or $G(O_m)$, $m \geq 1$, we write $g(0)$ for its image in $G(O_1) = G(F)$.

**Lemma 1.1.** For any element $g \in G(O)$ such that $g(0) = 1$ and any $n \in \mathbb{N}$, prime to char $F$, there exists $f \in G(O)$ such that $f(0) = 1$ and $g = f^n$. 

Proof. It suffices to show that any element \( h \in G(O_m), m \geq 1 \), such that \( h(0) = 1 \) and the image of \( g \) in \( G(O_m) \) equals \( h^m \), can be lifted to an element \( h_1 \in G(O_{m+1}) \) such that the image of \( g \) in \( G(O_{m+1}) \) equals \( h_1^n \). Consider the following exact sequence

\[
1 \longrightarrow \text{Lie}(G) \longrightarrow G(O_{m+1}) \longrightarrow G(O_m) \longrightarrow 1.
\]

The conjugation action of \( h \) on \( \text{Lie}(G) \) is given by the adjoint transformation \( \text{Ad}(h(0)) \), which is trivial since \( h(0) = 1 \). Hence any lifting of \( h \) to \( G(O_{m+1}) \) centralizes \( \text{Lie}(G) \), therefore the existence of the lifting we need follows from the fact that the group \( \text{Lie}(G) \) is \( n \)-divisible (being a vector space over \( F \)). \( \square \)

1.2. Cycle modules. A cycle module over a field \( F \) is an object function \( M : \text{fields} \to \text{Ab} \) together with a \( \mathbb{Z} \)-grading \( M = \bigoplus_n M_n \) and with some data and rules (cf. [24]). The data includes a graded module structure on \( M \) under the Milnor ring \( K_*(F) \), a degree 0 homomorphism \( \alpha : M(E) \to M(K) \) for any \( \alpha : E \to K \) in \( \text{fields} \) and also a degree \(-1\) residue homomorphism \( \partial_v : M(E) \to M(K) \) for a valuation \( v \) on \( E \) (here \( K(v) \) is the residue field).

We will always assume that \( M_n = 0 \) for \( n < 0 \).

Example 1.2. (cf. [24, Rem. 1.11]) Let \( N \) be a discrete torsion module over the absolute Galois group of a field \( F \). Set \( N[i] = \lim_{\rightarrow} (aN) \otimes \mu_n^{\otimes i} \), where \( \mu_n \) is the group of \( n \)-th roots of unity. Define a cohomological cycle module \( M_* = H^*[N] \) by

\[
M_d(L) = H^d[N](L) = H^d(L,N[d]).
\]

Let \( M \) be a cycle module over \( F \) and let \( K \) be a discrete valuation field over \( F \) with valuation \( v \) trivial on \( F \) and residue field \( L \). A choice of a prime element \( \pi \in K \) determines a specialization homomorphism \( s_{\pi} : M(K) \to M(L) \) (cf. [24, p.329]).

Let \( X \) be an algebraic variety over \( F \). For any \( i \) the homology group of the complex

\[
\bigoplus_{x \in X^{i-1}} M_{d-i+1}(F(x)) \xrightarrow{\partial} \bigoplus_{x \in X^{i}} M_{d-i}(F(x)) \xrightarrow{\partial} \bigoplus_{x \in X^{i+1}} M_{d-i-1}(F(x))
\]

we denote \( A^i(X, M_d) \). In particular, \( A^0(\text{Spec}(F), M_d) = M_d(F) \).

A morphism \( f : X \to Y \) of smooth varieties over \( F \) induces inverse image homomorphism (pull-back)(cf. [24, Sec.12])

\[
p^* : A^i(Y, M_d) \to A^i(X, M_d).
\]

Lemma 1.3. Let \( X \) be a smooth algebraic variety over a field \( F \), having an \( F \)-point. Then the natural homomorphism \( M(F) \to M(F(X)) \) is an injection.

Proof. The homomorphism \( M(F) \to M(F(X)) \) factors as

\[
M(F) \longrightarrow A^0(X, M_d) \hookrightarrow M(F(X)).
\]

The first homomorphism splits by the pull-back with respect to any \( F \)-rational point of \( X \). \( \square \)
1.3. The simplicial scheme $BG$. For an algebraic group $G$ over a field $F$
we consider the simplicial scheme $BG$ with $BG_0 = G^n$ and the face maps
$\partial_i : G^n \rightarrow G^{n-1}$, $i = 0, 1, \ldots, n$, given by the formulae

$$\partial_i(g_1, g_2, \ldots, g_n) = \begin{cases} (g_2, g_3, \ldots, g_n) & \text{if } i = 0, \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & \text{if } 0 < i < n, \\ (g_1, g_2, \ldots, g_{n-1}) & \text{if } i = n. \end{cases}$$

Let $\mathcal{A}$ be a full subcategory in $F$-groups closed under products (for example,
a subcategory of reductive groups) and let $F : \mathcal{A} \rightarrow \mathfrak{Ab}$ be a contravariant
functor. The homology groups of the complex $C^n = F(G^n)$ with the differentials

$$d_{n-1} : C^{n-1} \rightarrow C^n, \quad d_{n-1} = \sum_{i=0}^{i=n} (-1)^i F(\partial_i)$$
we denote $H^*(BG, F)$.

Clearly, $H^0(BG, F) = F(1)$. The group $H^1(BG, F)$ coincides with the
subgroup in $F(G)$ consisting of all elements $x$ such that

$$(1) \quad F(p_1)(x) + F(p_2)(x) = F(m)(x),$$
where $p_i : G \times G \rightarrow G$ are two projections and $m : G \times G \rightarrow G$ is the
multiplicative morphism. We simply denote this subgroup $F(G)_{mult}$ and call the
multiplicative part of $F(G)$.

Example 1.4. Let $A$ be an abelian group. Consider the functor $F(G) = Map(G(F), A)$. Then $H^n(BG, F) = H^n(G(F), A)$ the cohomology group of
$G(F)$ with coefficients in $A$ considered as a trivial $G(F)$-module. In particular,$F(G)_{mult} = \text{Hom}(G(F), A)$.

We write $F_{mult} : \mathcal{A} \rightarrow \mathfrak{Ab}$ for the functor given by $F_{mult}(G) = F(G)_{mult}$.
A contravariant functor $F : \mathcal{A} \rightarrow \mathfrak{Ab}$ is called constant, if $F(\alpha) : F(H) \rightarrow F(G)$ is an isomorphism for any group homomorphism $\alpha : G \rightarrow H$ in $\mathcal{A}$
and called additive if for any pair of algebraic groups $G$ and $H$ in $\mathcal{A}$, the
homomorphism

$$F(p_1) \oplus F(p_2) : F(G) \oplus F(H) \rightarrow F(G \times H)$$
is an isomorphism.

Lemma 1.5. (i) The functor $F_{mult}$ is additive for any $F$.

(ii) If $F$ is additive, then $F_{mult} = F$.

Proof. (i) Let $G$ and $H$ be two groups in $\mathcal{A}$. We have to show that the map

$$F(p_1) \oplus F(p_2) : F(G)_{mult} \oplus F(H)_{mult} \rightarrow F(G \times H)_{mult}$$
is an isomorphism.

Consider the embeddings $i : G \rightarrow G \times H$, $i(g) = (g, 1)$ and $j : H \rightarrow G \times H$,
$j(h) = (1, h)$. The composites $p_2 \circ i$ and $p_1 \circ j$ factor through a trivial group.
Since $F(1)_{mult} = 0$, it follows that $F(i) \circ F(p_2) = 0 = F(j) \circ F(p_1)$. Hence the
map $\mathcal{F}(p_1) \oplus \mathcal{F}(p_2)$ is a split injection. It remains to show that the restriction of $\ker \mathcal{F}(i) \cap \ker \mathcal{F}(j)$ on $\mathcal{F}(G \times H)_{\text{mult}}$ is trivial.

Consider the following commutative diagram:
\[
\begin{array}{ccc}
G \times H & \xrightarrow{p_1} & G \\
\downarrow{k} & & \downarrow{i} \\
G \times H \times G \times H & \xrightarrow{q_1} & G \times H,
\end{array}
\]
where $q_1$ is the first projection and $k(g, h) = (g, 1, 1, h)$. For any $x \in \ker \mathcal{F}(i)$, $\mathcal{F}(k) \circ \mathcal{F}(q_1)(x) = \mathcal{F}(p_1) \circ \mathcal{F}(i)(x) = 0$. Similarly, $\mathcal{F}(k) \circ \mathcal{F}(q_2)(x) = 0$, where $q_2$ is the second projection of $G \times H \times G \times H$ onto $G \times H$. If $x \in \mathcal{F}(G \times H)_{\text{mult}}$, then $\mathcal{F}(q_1)(x) + \mathcal{F}(q_2)(x) = \mathcal{F}(m)(x)$, where $m$ is the multiplication morphism for $G \times H$. Finally, $m \circ k = \text{id}_{G \times H}$, hence
\[
x = \mathcal{F}(\text{id})(x) = \mathcal{F}(k) \circ \mathcal{F}(m)(x) = \mathcal{F}(k) \circ \mathcal{F}(q_1)(x) + \mathcal{F}(k) \circ \mathcal{F}(q_2)(x) = 0.
\]

(ii) Now let $\mathcal{F}$ be additive. In the notation of the first part of the proof take $H = G$. Since $m \circ i = \text{id}_G = m \circ j$, the composite $(\mathcal{F}(i), \mathcal{F}(j)) \circ \mathcal{F}(m)$ is the diagonal map $\mathcal{F}(G) \to \mathcal{F}(G) \oplus \mathcal{F}(G)$. But $(\mathcal{F}(i), \mathcal{F}(j))$ is the inverse of $\mathcal{F}(p_1) \oplus \mathcal{F}(p_2)$. Hence $\mathcal{F}(p_1) + \mathcal{F}(p_2) = \mathcal{F}(m)$ and therefore $\mathcal{F}(G)_{\text{mult}} = \mathcal{F}(G)$.

\[\text{Corollary 1.6.} \quad \text{A functor } \mathcal{F} \text{ is additive if and only if } \mathcal{F}_{\text{mult}} = \mathcal{F}. \quad \Box\]

\[\text{Lemma 1.7.} \quad \text{[8, Lemma 4.5]} \quad \text{Let } \mathcal{F} : \mathcal{A} \to \mathfrak{Ab} \text{ be a functor. Then for any } G \text{ in } \mathcal{A},
\]
(i) If $\mathcal{F}$ is constant, then

\[
H^i(BG, \mathcal{F}) = \begin{cases} 
\mathcal{F}(1) & \text{if } i = 0, \\
0 & \text{if } i \neq 0.
\end{cases}
\]

(ii) If $\mathcal{F}$ is additive, then

\[
H^i(BG, \mathcal{F}) = \begin{cases} 
\mathcal{F}(G) & \text{if } i = 1, \\
0 & \text{if } i \neq 1. \quad \Box
\end{cases}
\]

\[\text{Corollary 1.8.} \quad \text{Let } 0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0 \text{ be an exact sequence of functors. If } \mathcal{F}_1 \text{ and } \mathcal{F}_3 \text{ are additive functors on } \mathcal{A}, \text{ then } \mathcal{F}_2 \text{ is also additive on } \mathcal{A}. \quad \Box\]

Let $u : 1 \to G$ be the unit morphism. For any functor $\mathcal{F} : \mathcal{A} \to \mathfrak{Ab}$ we write $\bar{\mathcal{F}}(G)$ for the kernel of $\mathcal{F}(u) : \mathcal{F}(G) \to \mathcal{F}(1)$.

\[\text{Lemma 1.9.} \quad \mathcal{F}(G)_{\text{mult}} \subset \bar{\mathcal{F}}(G).
\]

\[\text{Proof.} \quad \text{Let } u' : 1 \to G \times G \text{ be the unit morphism. Applying } \mathcal{F}(u') \text{ to both sides of (1), we get } 2\mathcal{F}(u')(x) = \mathcal{F}(u)(x). \quad \Box\]

2. Definition of an Invariant

In this section we give definition of invariant of an algebraic group and compute the group of invariants in simple cases.
2.1. Definition. Let $G$ be an algebraic group over a field $F$ and $M$ be a $(\mathbb{Z}-$graded) cycle module over $F$. For any $d \in \mathbb{Z}$, we consider $M_d$ as a functor from $\textit{F-fields}$ to $\textit{Groups}$. An invariant of $G$ in $M$ of dimension $d$ is a natural transformation of functors $G \to M_d$ from the category $\textit{F-fields}$ to $\textit{Groups}$.

All invariants of $G$ in $M$ of degree $d$ form an abelian group, which we denote $\text{Inv}^d(G, M)$. For any $u \in \text{Inv}^d(G, M)$ and any $L \in \textit{F-fields}$, we write $u_L$ for the corresponding homomorphism $\text{Inv}^d(L) \to M_d(L)$.

An algebraic group homomorphism $G \to H$ induces a homomorphism

$$
\text{Inv}^d(H, M) \to \text{Inv}^d(G, M)
$$

making $G \mapsto \text{Inv}^d(G, M)$ a (contravariant) functor from $\textit{F-groups}$ to $\textit{Ab}$.

Clearly, for two groups $G$ and $H$ over $F$ there is a canonical isomorphism

$$
\text{Inv}^d(G \times H, M) \cong \text{Inv}^d(G, M) \oplus \text{Inv}^d(H, M),
$$

i.e. the functor $\text{Inv}^d(\ast, M)$ is additive.

Suppose that $G$ is a connected group. Set $K = F(G)$, the function field of $G$ and denote $\xi \in G(K)$ the generic point of $G$. For any invariant $u \in \text{Inv}^d(G, M)$, $u_K(\xi)$ is an element of $M_d(K)$, so that we have a homomorphism

$$
\theta : \text{Inv}^d(G, M) \to M_d(K), \quad u \mapsto u_K(\xi).
$$

Lemma 2.1. The image of $\theta$ is contained in $A^0(G, M_d)$.

Proof. Let $P = F(G \times G)$ and let $\bar{p}_1, \bar{p}_2$ and $\bar{m}$ be the $F$-homomorphisms of fields $K \to P$ induced respectively by the first, the second projections and by the multiplication morphism $p_1, p_2, m : G \times G \to G$. Consider the elements $\xi_1 = G(\bar{p}_1)(\xi)$, $\xi_2 = G(\bar{p}_2)(\xi)$ and $\xi' = G(\bar{m})(\xi)$ in the group $G(P)$. Clearly, $\xi' = \xi_1 \cdot \xi_2$.

Let $u \in \text{Inv}^d(G, M)$ be any invariant. We have

$$
u_P(\xi') = u_P(\xi_1) + u_P(\xi_2) \in M_d(P).
$$

We need to show that $\partial_h(u_L(\xi)) = 0 \in M_{d-1}(F(g))$ for any point $g$ in $G$ of codimension 1. Let $h$ be the point of codimension 1 in $G \times G$ such that $\{h\} = \{g\} \times G$. The projection $p_2$ takes $h$ to the generic point of $G$. Hence by the rule R3c in [24, p.329], $\partial_h(u_P(\xi_2)) = 0 \in M_{d-1}(F(h))$. By the same reasoning, $\partial_h(u_P(\xi')) = 0$. Hence,

$$
\partial_h(u_P(\xi_1)) = \partial_h(u_P(\xi')) - \partial_h(u_P(\xi_2)) = 0 \in M_{d-1}(F(h)).
$$

Let $k : F(g) \to F(h)$ be the field homomorphism induced by the projection $p_1$. By the rule R3a in [24, p.329], applied to the field extension $\bar{p}_1 : K \to P$, we have

$$
k_*(\partial_g(u_K(\xi))) = \partial_h(p_1*(u_K(\xi))) = \partial_h(u_P(\xi_1)) = 0 \in M_{d-1}(F(h)).
$$

The field $F(h)$ is isomorphic to $F(g)/(G)$. By Lemma 1.3, the homomorphism $k_* : M_{d-1}(F(g)) \to M_{d-1}(F(h))$ is injective and hence $\partial_g(u_L(\xi)) = 0 \in M_{d-1}(F(g))$. $\square$
We say that a cycle module \( M \) is of bounded exponent, if there exists \( n \in \mathbb{N} \), prime to \( \text{char} \, F \), such that \( nM = 0 \).

**Lemma 2.2.** Let \( M \) be a cycle module of bounded exponent and let \( g_1 \) and \( g_2 \) be two points of \( G \) such that \( g_2 \) is regular and of codimension 1 in \( \{ g_1 \} \). Suppose that for an invariant \( u \in \text{Inv}^d(G, M) \) we have \( u_{F(g_1)}(g_1) = 0 \in M_d(F(g_1)) \). Then \( u_{F(g_2)}(g_2) = 0 \in M_d(F(g_2)) \).

**Proof.** Denote \( A \) the local ring of the point \( g_2 \) in the variety \( \{ g_1 \} \). By assumption, \( A \) is a discrete valuation ring with fraction field \( F(g_1) \) and residue field \( F(g_2) \). Denote \( l : A \to F(g_2) \) the natural surjection. The image of the generic element under \( G(F[G]) \to G(A) \), induced by the natural ring homomorphism \( F[G] \to A \), equals \( g_1 \in G(A) \subset G(F(g_1)) \). Clearly \( G(1)(g_1) = g_2 \).

We write \( \tilde{A} \) for the completion of \( A \) with respect to the natural discrete valuation. By a theorem of Cohen (cf. [30, Ch. VIII, Th. 27]), there is a section \( p : F(g_2) \to \tilde{A} \) of the natural surjection \( q : A \to F(g_2) \) and \( \tilde{A} \simeq F(g_2)[[t]] \). Set \( \tilde{g}_1 = G(p)(g_2) \in G(\tilde{A}) \). Then

\[
G(q)(g_1) = G(l)(g_1) = g_2 = G(q)(\tilde{g}_1),
\]

hence \( g_1 \cdot \tilde{g}_1^{-1} \) belongs to the kernel of \( G(q) \).

Choose \( n \in \mathbb{N} \), prime to \( \text{char} \, F \), such that \( nM_d = 0 \). By Lemma 1.1, \( g_1 \cdot \tilde{g}_1^{-1} = h^m \) for some \( h \in G(\tilde{A}) \).

Let \( L \) be the fraction field of \( \tilde{A} \) and let \( i \) be the embedding of \( F(g_2) \) into \( L \). We have

\[
0 = u_L(g_1) = u_L(h^m \cdot \tilde{g}_1) = u_L(\tilde{g}_1) = i_* (u_{F(g_2)}(g_2)) \in M_d(L).
\]

Since \( L \simeq F(g_2)[[t]] \), the specialization homomorphism \( s_v^t \) for the discrete valuation \( v \) on \( L \) splits \( i_* \), hence \( i_* \) is injective and \( u_{F(g_2)}(g_2) = 0 \). \( \square \)

The following statement is useful for computations of invariants.

**Theorem 2.3.** Let \( G \) be a connected algebraic group over \( F \) and \( M \) be a cycle module over \( F \) of bounded exponent. Then the map \( \theta \) induces an isomorphism

\[
\text{Inv}^d(G, M) \to \text{A}^0(G, M_d)_{\text{mult}}.
\]

**Proof.** It follows from the proof of Lemma 2.1 that the image of \( \theta \) belongs to \( \text{A}^0(G, M)_{\text{mult}} \).

**Injectivity.** Assume that for \( u \in \text{Inv}^d(G, M) \) we have \( u_{F(G)}(\xi) = 0 \). For a field extension \( L/F \) take any \( h \in G(L) \), i.e. a morphism \( h : \text{Spec}(L) \to G \). We have to show that \( u_L(h) = 0 \). Denote \( g \in G \) the only point in the image of \( h \). There is a sequence of points \( \xi = g_1, g_2, \ldots, g_m = g \) such that \( g_{i+1} \) is regular and of codimension 1 in \( \{ g_i \} \) for all \( i = 1, 2, \ldots, m - 1 \). By Lemma 2.2, \( u_{F(g)}(g) = 0 \). The element \( h \) is the image of \( g \) under \( G(F(g)) \to G(L) \), induced by the natural homomorphism \( F(g) \to L \), hence \( u_L(h) = 0 \), being the image of \( u_{F(g)}(g) \) under \( M_d(F(g)) \to M_d(L) \).
**Surjectivity.** Let \( a \in A^0(G, M_d)_{\text{mult}} \). For any \( L \in F\text{-fields} \) we define a homomorphism \( u_L : G(L) \to M_d(L) \) by the formula \( u_L(g) = g^*(a) \), where 
\[
g^* : A^0(G, M_d) \to A^0(\text{Spec}(L), M_d) = M_d(L)
\]
is the inverse image map with respect to \( g : \text{Spec}(L) \to G \).

We show first that \( u_L \) is a homomorphism. Let \( g_1, g_2 \in G(L) \). Denote \( g \) the morphism \( (g_1, g_2) : \text{Spec}(L) \to G \times G \). We have \( p_i \circ g = g_i \) and \( m \circ g = g_1 g_2 \).

By definition of the multiplicative part of \( A^0(G, M_d) \),
\[
p^*_1(a) + p^*_2(a) = m^*(a) \in A^0(G \times G, M_d).
\]

Hence
\[
u_L(g_1 g_2) = (g_1 g_2)^*(a) = g^*(m^*(a)) = g^*(p^*_1(a) + p^*_2(a)) = g_1^*(a) + g_2^*(a) = u_L(g_1) + u_L(g_2),
\]
i.e. \( u_L \) is a homomorphism.

Let \( \alpha : L \to E \) be a \( F \)-homomorphism of fields. Denote \( f : \text{Spec}(E) \to \text{Spec}(F) \) the corresponding morphism. Then for any \( g \in G(L) \) one has
\[
(\alpha \circ u_L)(g) = \alpha_* (g^*(a)) = f^*(g^*(a)) = (gf)^*(a) = u_E(gf) = (u_E \circ G(\alpha))(g),
\]
i.e. the following diagram is commutative:
\[
\begin{array}{ccc}
G(L) & \xrightarrow{u_L} & M(L) \\
\downarrow G(\alpha) & & \downarrow \alpha_* \\
G(E) & \xrightarrow{u_E} & M(E).
\end{array}
\]
Hence \( u \) is a functor from \( F\text{-fields} \) to \( \text{Groups} \), i.e. \( u \in \text{Inv}^d(G, M) \).

Finally it suffices to show that \( \theta(u) = a \). By definition of \( \theta \), \( \theta(u) = u_{F(G)}(\xi) = \xi^*(a) \) and the latter is equal to \( a \), since the inverse image homomorphism
\[
\xi^* : A^0(G, M_d) \to A^0(\text{Spec}(F(G), M_d) = M_d(F(G))
\]
is the natural inclusion by [24, Cor. 12.4]. \( \square \)

2.2. **Invariants of unipotent groups.** We show that unipotent groups have no nontrivial invariants.

**Proposition 2.4.** Let \( G \) be a unipotent group. Then \( \text{Inv}^d(G, M) = 0 \) for any \( M \) of bounded exponent.

**Proof.** For the proof we may assume that \( G \) is connected, since \( G/G^0 \) is a \( p \)-group, \( p = \text{char } F \) and the cycle module \( M \) has exponent prime to \( p \). If \( F \) is a perfect field, then by [6, 15.13], the variety of \( G \) is isomorphic to an affine space, hence \( \bar{H}^0(G, M_d) = 0 \) and \( \text{Inv}^d(G, M) = 0 \) by Theorem 2.3 and Lemma 1.9. In the general case consider the perfect field \( L = F^{p^{-\infty}} \), where \( p = \text{char } F \). Since the natural map \( M(F) \to M(L) \) is injective (exponent of \( M \) is prime to \( p \)), and any invariant is trivial on \( G(L) \), it follows that it is trivial on \( G(F) \subset G(L) \). \( \square \)
Let $G$ be a connected algebraic group over a perfect field $F$. The unipotent radical $U$ of $G$ is then defined over $F$. Denote $G'$ the reductive factor group $G/U$. For any $L \in F\text{-alg}$ the natural homomorphism $G(L) \to G'(L)$ is surjective since $H^1(L, U) = 1$ (cf. [29, 18.2]). Thus, by Proposition 2.4, the natural homomorphism

$\text{Inv}^d(G, M) \to \text{Inv}^d(G', M)$

is an isomorphism. This reduces computation of invariants to the case of reductive groups.

2.3. Invariants of split reductive groups. Let $G$ be a split reductive group over a field $F$. A character $\chi \in G^*$ and an element $x \in M_{d-1}(F)$ define an invariant $u^{\chi, x}$ as follows. For any $L \in F$-fields and any $g \in G(L)$, we set

$u^{\chi, x}_L(g) = i_*\{x\} \cdot \{\chi(g)\} \in M_d(L)$

where $i : F \to L$ is the natural homomorphism.

**Proposition 2.5.** Assume that the derived subgroup $G'$ in $G$ is simply connected. Then the map $G^* \otimes M_{d-1}(F) \to \text{Inv}^d(G, M)$, $\chi \otimes x \to u^{\chi, x}$, is an isomorphism.

**Proof.** Consider first the case $G = \mathbb{G}_m$. By [24, Prop. 2.2],

$A^0(\mathbb{G}_m, M_d) = M_d(F) \oplus M_{d-1}(F) \cdot \{t\}$,

hence $\tilde{A}^0(\mathbb{G}_m, M_d) = M_{d-1}(F) \cdot \{t\}$. The inclusion

$\text{Inv}^d(G, M) \simeq A^0(\mathbb{G}_m, M_d)_{\text{mult}} \hookrightarrow \tilde{A}^0(\mathbb{G}_m, M_d) \simeq M_{d-1}(F)$

is then the inverse to the homomorphism in question. Since the functor $G \mapsto \text{Inv}^d(G, M)$ is additive, the results holds for any split torus $G$.

Now consider an arbitrary split reductive group $G$ over $F$ with the simply connected derived subgroup $G'$. Denote $S$ the torus $G/G'$. By [8, 3.20], the natural homomorphism

$A^0(S, M_d) \to A^0(G, M_d)$

is an isomorphism. Hence the left vertical homomorphism in the diagram

$A^0(S, M_d)_{\text{mult}} \sim \text{Inv}^d(S, M) \hookrightarrow S^* \otimes M_{d-1}(F)$

is an isomorphism. By Theorem 2.3, the middle vertical map is also an isomorphism. Then we are reduced to the considered case of a split algebraic tori, since $S^* \simeq G^*$.

**Corollary 2.6.** Let $G$ be a split simply connected semisimple group. Then $\text{Inv}^d(G, M) = 0$, i.e. $G$ has no nontrivial invariants.

**Example 2.7.** The determinant map defines isomorphisms

$\text{Inv}^d(\text{GL}_n, M) \sim \text{Inv}^d(\mathbb{G}_m, M) \simeq M_{d-1}(F)$. 

3. COHOMOLOGICAL INVARIANTS

A cohomological invariant of an algebraic group $G$ defined over a field $F$ is an invariant in a cycle module $H^*[N]$, where $N$ is a torsion module over the absolute Galois group $\Gamma$ of $F$.

For a variety $X$ over $F$ and a torsion Galois module $N$ we set

$$A^i(X, H^j(N)) = A^i(X, H^j[N[-j]]).$$

There is a Bloch-Ogus spectral sequence (cf. [5])

$$E_2^{p,q} = H^p(X, H^q(N)) \Rightarrow H^{p+q}_{\text{et}}(X, N).$$

3.1. Invariants of dimension 0. Let module $N$ be of bounded exponent, i.e. $nN = 0$ for some $n$ prime to char $F$. The corresponding cycle module $H^*[N]$ is also of bounded exponent.

We describe all invariants of dimension 0 of an algebraic group $G$. Denote $G^0$ the connected component of the unity in $G$ and set $\hat{G} = G/G^0$, so that $\hat{G}$ is an étale group (cf. [29]). The absolute Galois group $\Gamma$ of $F$ acts naturally on the group $\hat{G}(F_{\text{sep}})$. Let $f : \hat{G}(F_{\text{sep}}) \to N$ be a homomorphism of $\Gamma$-modules. We write $u^f$ for the composite

$$G(L) \to \hat{G}(L) = \hat{G}(F_{\text{sep}})^\Gamma \xrightarrow{f} N^\Gamma = H^0(F, N).$$

Similarly, one defines $u_L^f$ for any $L \in F$-fields. Clearly, $u^f$ is an invariant of dimension 0.

**Theorem 3.1.** For a $\Gamma$-module $N$ of bounded exponent, the map

$$\alpha : \text{Hom}_\Gamma(\hat{G}(F_{\text{sep}}), N) \to \text{Inv}^0(G, H^*[N]),$$

$\alpha(f) = u^f$, is an isomorphism.

**Proof.** Injectivity. Assume that $\alpha(f) = u^f = 0$. If $L = F_{\text{sep}}$, then the map $G(L) \to \hat{G}(L)$ is surjective, hence $f = 0$.

Surjectivity. Choose $n \in \mathbb{N}$ such that $(n, \text{char } F) = 1$ and $nN = 0$. The group $G^0$ is connected, hence $G^0(L)$ is $n$-divisible for $L = F_{\text{sep}}$. Therefore, for any invariant $u \in \text{Inv}^0(G, H^*[N])$, the map

$$u_L : G(L) \to H^0(L, N) = N$$

factors through a homomorphism $f : \hat{G}(L) = G(L)/G^0(L) \to N$. Clearly, $f$ is $\Gamma$-equivariant and $\alpha(f) = u$. □

**Corollary 3.2.** If $G$ is connected, then $\text{Inv}^0(G, H^*[N]) = 0$. □

3.2. Invariants of dimension 1. Let $G$ be a connected group over $F$, and let $N$ be a Galois module of bounded exponent. The Bloch-Ogus spectral sequence (2) for the module $N[1]$ shows that the natural homomorphism

$$H^1_{\text{et}}(G, N[1]) \to A^0(G, H^1(N[1])) = A^0(G, H^1[N])$$

is an isomorphism.

Theorem 2.3 can be then reformulated as follows.
Proposition 3.3. There is a natural isomorphism
\[ \text{Inv}^1(G, H^*[N]) \sim H^1_{et}(G, N[1])_{\text{mult}}. \]

The Hochschild-Serre spectral sequence
\[ H^p(F, H^q(G_{\text{sep}}, N[1])) \Rightarrow H^{p+q}(G, N[1]) \]
induces an exact sequence
\[ 0 \longrightarrow H^1(F, N[1]) \longrightarrow H^1_{et}(G, N[1]) \longrightarrow H^0(F, H^1_{et}(G_{\text{sep}}, N[1])) \longrightarrow H^2(F, N[1]) \longrightarrow H^2_{et}(G, N[1]). \]
The latter homomorphism is a split injection, hence
\[ H^1_{et}(G, N[1]) \simeq H^0(F, H^1_{et}(G_{\text{sep}}, N[1])). \]

The Kummer exact sequence for \( n \) prime to char \( F \),
\[ 1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \overset{n}{\longrightarrow} \mathbb{G}_m \longrightarrow 1 \]
gives rise to the following exact sequence:
\[ 0 \longrightarrow F[G]^\times/F[G]^\times_n \longrightarrow H^1_{et}(G, \mu_n) \longrightarrow n \, \text{Pic}(G) \longrightarrow 0. \]
By [22], \( F[G]^\times = F^\times \oplus G^\times \). Hence we have the following exact sequence
\[ 0 \longrightarrow G^*/G^*n \longrightarrow \tilde{H}^1_{et}(G, \mu_n) \longrightarrow n \, \text{Pic}(G) \longrightarrow 0. \]

If \( G \) is reductive, by [25, Lemma 6.9(i)], the functor \( G \mapsto \text{Pic}(G_{\text{sep}}) \) is additive on the category of reductive groups. The functor \( G \mapsto G^*_{\text{sep}} \) is clearly also additive. Hence, by Corollary 1.8, the functor \( G \mapsto \tilde{H}^1_{et}(G, \mu_n) \) is additive. Over \( F_{\text{sep}} \) the module \( N[1] \) is isomorphic to a direct sum of \( \mu_n \)'s for various \( n \), hence the functor \( H^1_{et}(G_{\text{sep}}, N[1]) \) is also additive. By (3), \( \tilde{H}^1_{et}(G, N[1]) \) is an additive functor. Lemma 1.5 then gives
\[ \tilde{H}^1_{et}(G, N[1]) = H^1_{et}(G, N[1])_{\text{mult}} = H^1_{et}(G, N[1])_{\text{mult}}. \]

Denote \( \text{Pic}^*(G_{\text{sep}}) \) the dual Galois module
\[ \text{Hom}(\text{Pic}(G_{\text{sep}}), F^\times_{\text{sep}}). \]
Suppose that \( N \) is a free \( \mathbb{Z}/n\mathbb{Z} \)-module. Tensoring by \( N \) the sequence (4) over \( F_{\text{sep}} \), we get an exact sequence
\[ 0 \longrightarrow G^*_{\text{sep}} \otimes N \longrightarrow \tilde{H}^1_{et}(G_{\text{sep}}, N[1]) \longrightarrow \text{Hom}(\text{Pic}^*(G_{\text{sep}}), N[1]) \longrightarrow 0. \]
Since any Galois module of bounded exponent is a direct sum of free \( \mathbb{Z}/n\mathbb{Z} \)-modules for all \( n \), this sequence exists for any \( N \) of bounded exponent. Taking cohomology groups, we get in view of (3)
\[ 0 \longrightarrow H^0(F, G^*_{\text{sep}} \otimes N) \longrightarrow \tilde{H}^1_{et}(G, N[1]) \longrightarrow \text{Hom}(\text{Pic}^*(G_{\text{sep}}), N[1]) \longrightarrow H^1(F, G^*_{\text{sep}} \otimes N). \]
Proposition 3.3 and (5) then give
Theorem 3.4. Let $G$ be a reductive group over $F$, $N$ be a Galois module of bounded exponent. Then there is an exact sequence

$$0 \rightarrow H^0(F, G^\text{sep} \otimes N) \rightarrow \text{Inv}^1(G, H^*[N]) \rightarrow \text{Hom}_F(\text{Pic}^*(G^\text{sep}), N[1]) \rightarrow H^1(F, G^\text{sep} \otimes N).$$

Corollary 3.5. If $G$ is a semisimple group, $C$ is the kernel of the universal cover $\tilde{G} \rightarrow G$, then

$$\text{Inv}^1(G, H^*[N]) \simeq \text{Hom}_F(C(F^\text{sep}), N[1]).$$

In particular, $\text{Inv}^1(G, H^*[\mathbb{Z}/n\mathbb{Z}]) \simeq nC^*$. 

Proof. There is a natural isomorphism $\text{Pic}(G^\text{sep}) \simeq C^*_\text{sep}$ (cf. [25, Lemma 6.9(iii)]), so that $\text{Pic}^*(G^\text{sep}) = C(F^\text{sep})$. □

Remark 3.6. Assume that the order of $C$ is prime to $\text{char } F$. Then Corollary 3.5 shows that the invariant $G(L) \rightarrow H^1(L, C)$, induced by the connecting homomorphism with respect to the exact sequence

$$1 \rightarrow C \rightarrow \tilde{G} \rightarrow G \rightarrow 1,$$

is the universal one.

Corollary 3.7. If $T$ is an algebraic torus, then

$$\text{Inv}^1(T, H^*[N]) \simeq H^0(F, T^\text{sep} \otimes N).$$

In particular, $\text{Inv}^1(T, H^*[\mathbb{Z}/n\mathbb{Z}]) \simeq H^0(F, T^\text{sep}/nT^\text{sep})$. □

Remark 3.8. For a commutative algebraic group $A$ the isomorphism classes of all central extensions of $G$ by $A$ form an abelian group $\text{Ext}(G, A)$. The Kummer sequence induces the following exact sequence

$$0 \rightarrow G^*/G^*n \rightarrow \text{Ext}(G, \mu_n) \rightarrow n\text{Ext}(G, \mathbb{G}_m) \rightarrow 0.$$ 

By [16, Lemma 1.6], the natural homomorphism $\text{Ext}(G, \mathbb{G}_m) \rightarrow \text{Pic}(G)$ is an isomorphism. Hence, in view of (4), the natural map $\text{Ext}(G, \mu_n) \rightarrow \check{H}^1_G(G, \mu_n)$, taking the class of an extension $1 \rightarrow \mu_n \rightarrow G' \rightarrow G \rightarrow 1$ to the class of $\mu_n$-torsor $G'$ over $G$, is also an isomorphism. The extension $G'$ of $G$ by $\mu_n$ induces an invariant in $\text{Inv}^1(G, H^*[\mathbb{Z}/n\mathbb{Z}])$ via the connecting homomorphism with respect to the exact sequence. Hence

$$\text{Inv}^1(G, H^*[\mathbb{Z}/n\mathbb{Z}]) \simeq \text{Ext}(G, \mu_n).$$

Example 3.9. Let $2n$ be prime to $\text{char } F$ and let $G$ be an adjoint semisimple group of type $^2A_{n-1}$, i.e. $G$ is the projective unitary group $\text{PGU}(B, \tau)$, where $B$ is a central simple algebra of dimension $n^2$ over a quadratic field extension $L/F$ with an involution $\tau$ on $B$ of the second kind, trivial on $F$. The kernel $C$ of the universal covering of $G$ is $\mu_n[L]$ (cf. [11]). The character group $C^*_\text{sep}$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ with the action of the absolute Galois group of $F$ via $\text{Gal}(L/F)$ taking $i+n\mathbb{Z}$ to $-i+n\mathbb{Z}$. Hence, if $n$ is odd, the group $C^*$ is trivial and $G$ has no nontrivial invariants in $H^*[\mathbb{Z}/k\mathbb{Z}]$ for any $k$ prime to $\text{char } F$. 

If $n$ is even, $n = 2m$, the group $C^*$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, so that there is only one nontrivial invariant $u \in \text{Inv}^1(G, H^*[\mathbb{Z}/2\mathbb{Z}]) \simeq 2C^*$. The homomorphism $u_F : G(F) \to H^1(F, \mu_2) = F^x/F^{x2}$ is defined as follows. An element of $G(F)$ is represented (modulo $L^x$) by an element $b \in B^x$ such that $b \cdot \tau(b) = \mu \in F^x$ (cf. [11]). The reduced norm $\beta = Nrd(b) \in L^x$ satisfies $N_{L/F}(\beta) = \mu^n$. Hence $N_{L/F}(\mu^{-m}\beta) = 1$ and by Hilbert’s Theorem 90, $\mu^{-m}\beta = \tau(\alpha)\alpha^{-1}$ for some $\alpha \in L^x$ uniquely determined modulo $F^x$. Hence, the class $N_{L/F}(\alpha)F^{x2} \in F^x/F^{x2}$ is well defined and

$$u_F(bL^x) = \mu^m \cdot N_{L/F}(\alpha)F^{x2} \in F^x/F^{x2} = H^1(F, \mu_2).$$

3.3. Invariants of dimension 2. We compute the group $\text{Inv}^2(G, H^*[\mu_2^{\otimes -1}])$ for any reductive group $G$ and any $n$ prime to $\text{char} F$. The direct limit of these groups for all $n$ prime to $\text{char} F$ we denote $\text{Inv}^2(G, H^*[\mu_2^{\otimes -1}])$. Clearly,

$$\text{Inv}^2(G, H^*[\mu_2^{\otimes -1}]) = \text{Inv}^2(G, H^*[\mu_2^{\otimes -1}]).$$

For a torsion abelian group $A$, we write $A'$ for the subgroup in $A$ of all elements of order prime to $\text{char} F$.

By Theorem 2.3, the group $\text{Inv}^2(G, H^*[\mu_2^{\otimes -1}])$ is isomorphic to

$$A^0(G, H^2[\mu_2^{\otimes -1}])_{\text{mult}} = A^0(G, H^2(\mu))_{\text{mult}}$$

which, in its turn, can be identified with the subgroup $\text{Br}(G)'_{\text{mult}}$ of the Brauer group $\text{Br}(G)$ by [7, Prop. 4.2.3(a)].

**Lemma 3.10.** Let $T$ be an algebraic torus over a separably closed field $F$. Then $\text{Br}(T)'_{\text{mult}} = 0$.

**Proof.** Since $F$ is separably closed, by Tsen’s Theorem, $\text{Br}(F(t))' = 0$, hence $\text{Br}(\mathbb{G}_m)'_{\text{mult}} = 0$. In the general case, $T \cong \mathbb{G}_m^n$ and the result follows by additivity from Lemma 1.5 applied to the functor $G \mapsto \text{Br}(G)'$.

This Lemma and Lemma 1.5 then imply

**Corollary 3.11.** Let $G$ be an algebraic group over a separably closed field $F$, $T$ be an algebraic torus over $F$. Then the embedding $G \to G \times T$, $g \mapsto (g, 1)$, induces an isomorphism $\text{Br}(G \times T)'_{\text{mult}} \sim \text{Br}(G)'_{\text{mult}}$.

**Proposition 3.12.** Let $G$ be a reductive group over a separably closed field $F$. Then $\text{Br}(G)'_{\text{mult}} = 0$.

**Proof.** Let $T$ be the connected center of $G$, $\overline{G} = G/T$. It follows from the proof of Lemma 6.12 in [25], that there is an exact sequence

$$0 \to \text{Br}(\overline{G}) \to \text{Br}(G) \xrightarrow{\beta} \text{Br}(G \times T),$$

where $\alpha$ is induced by the natural epimorphism $G \to \overline{G}$ and $\beta = \beta_1 - \beta_2$ with $\beta_1$ induced by the projection $G \times T \to G$ and $\beta_2$ induced by the multiplication morphism $G \times T \to G$. Since $T$ is a torus, by Corollary 3.11, $\beta_1$ and $\beta_2$ coincide on $\text{Br}(G)'_{\text{mult}}$. Hence, it suffices to show that $\text{Br}(\overline{G})'_{\text{mult}} = 0$, so that now we may assume that $G$ is a semisimple group.
By [9, Th. 4.1], the group \( \text{Br}(G)' \) is isomorphic to \( H^3(C^*, \mathbb{Z})' \) where \( C \) the kernel of the universal cover \( \tilde{G} \to G \). Thus, it suffices to show that for the functor \( \mathcal{F}(H) = H^3(H, \mathbb{Z}) \) the group \( \mathcal{F}(H)_{\text{mult}} \) is trivial for any finite abelian (constant) group \( H \). Since \( H \) is a product of cyclic groups and \( H^3(H, \mathbb{Z}) \) is trivial for a cyclic group \( H \), the result follows from Lemma 1.5.

Consider hypercohomology groups \( \mathbb{H}^n_{\text{et}}(BG, \mathbb{G}_m) \) (cf. [8]). There is a spectral sequence

\[
E_2^{p,q} = H^p(BG, H^q_{\text{et}}(*, \mathbb{G}_m)) \Rightarrow \mathbb{H}^{p+q}_{\text{et}}(BG, \mathbb{G}_m).
\]

Since \( H^0_{\text{et}}(G, \mathbb{G}_m) = F^\times \oplus G^* \), by Lemma 1.7,

\[
E_2^{p,0} = \begin{cases} 
F^\times & \text{if } p = 0, \\
G^* & \text{if } p = 1, \\
0 & \text{if } p \geq 2.
\end{cases}
\]

Since \( H^1_{\text{et}}(G, \mathbb{G}_m) = \text{Pic}(G) \) is an additive functor on the category of reductive groups, again by Lemma 1.7,

\[
E_2^{p,1} = \begin{cases} 
0 & \text{if } p = 0, \\
\text{Pic}(G) & \text{if } p = 1, \\
0 & \text{if } p \geq 2.
\end{cases}
\]

The group \( H^2_{\text{et}}(G, \mathbb{G}_m) \) is the Brauer group \( \text{Br}(G) \), hence

\[
E_2^{p,2} = \begin{cases} 
\text{Br}(F) & \text{if } p = 0, \\
\text{Br}(G)_{\text{mult}} & \text{if } p = 1.
\end{cases}
\]

Then the spectral sequence gives

\[
\mathbb{H}^n_{\text{et}}(BG, \mathbb{G}_m) = \begin{cases} 
0 & \text{if } n = 0, \\
G^* & \text{if } n = 1, \\
\text{Pic}(G) & \text{if } n = 2, \\
\text{Br}(G)_{\text{mult}} & \text{if } n = 3.
\end{cases}
\]

**Theorem 3.13.** Let \( G \) be a reductive group over a field \( F \). Then there is an exact sequence

\[
0 \to H^0(F, \text{Pic}(G_{\text{sep}}))' \to H^2(F, G^*_{\text{sep}})' \to \text{Inv}^2(G, H^* |_{\mu^{\otimes -1}}) \to H^1(F, \text{Pic}(G_{\text{sep}}))' \to H^3(F, G^*_{\text{sep}})'.
\]

**Proof.** Consider the Hochschild-Serre spectral sequence

\[
E_2^{p,q} = H^p(F, \tilde{\mathbb{H}}^n_{\text{et}}(BG_{\text{sep}}, \mathbb{G}_m)) \Rightarrow \tilde{\mathbb{H}}^n_{\text{et}}(BG, \mathbb{G}_m).
\]

We have

\[
E_2^{p,q} = \begin{cases} 
0 & \text{if } q = 0, \\
H^0(F, G^*_{\text{sep}}) & \text{if } q = 1, \\
H^q(F, \text{Pic}(G_{\text{sep}})) & \text{if } q = 2.
\end{cases}
\]
By Proposition 3.12, \((F^{0,3}_2)'/\subset \operatorname{Br}(\mathcal{G}_{\text{sep}})^\text{mult} = 0\), so the desired exact sequence is induced by the spectral sequence and the isomorphisms
\[
\operatorname{Inv}^2(G, H^*[{\mu_n}^\otimes^{-1}]) \cong \operatorname{Br}(G)^\text{mult} = \tilde{H}^3_{\text{et}}(BG, \mathbb{G}_m)'.
\]

**Corollary 3.14.** If \(G\) is a semisimple group, \(C\) is the kernel of the universal cover \(\tilde{G} \to G\), then
\[
\operatorname{Inv}^2(G, H^*[{\mu_n}^\otimes^{-1}]) \cong H^1(F, C_{\text{sep}}^*)' . \quad \square
\]

**Corollary 3.15.** Let \(T\) be an algebraic torus. The pairing
\[
T(F) \otimes H^2(F, T_{\text{sep}}^*) \to H^2(F, \mathbb{G}_m) = \operatorname{Br}(F)
\]
induces an isomorphism
\[
H^2(F, T_{\text{sep}}^*)' \cong \operatorname{Inv}^2(T, H^*[{\mu_n}^\otimes^{-1}]). \quad \square
\]

**Example 3.16.** Assume that \(\text{char } F \neq 2\). Let \(G\) be the special orthogonal group of type \(B\) or \(D\) over \(F\), i.e. \(G = \mathbf{O}_+(A, \sigma)\), where \(A\) is a central simple algebra over \(F\) with an orthogonal involution \(\sigma\). (If \(A\) splits, \(A \simeq \text{End}(V)\), the involution \(\sigma\) is adjoint with respect to a quadratic form \(q\) on \(V\), so that \(G\) is the special orthogonal group \(\mathbf{O}_+(V, q)\), cf. [11].) The kernel \(C\) of the universal covering of \(G\) is \(\mu_2\), so that \(C^*\) is isomorphic to \(\mathbb{Z}/2\mathbb{Z}\). Hence, \(\operatorname{Inv}^2(G, H^*[\mathbb{Z}/2\mathbb{Z}]) \cong H^1(F, \mathbb{Z}/2\mathbb{Z}) \cong F^\times/F^\times 2\). Let \(u\) be the invariant corresponding to an element \(a \in F^\times\). Then for any \(g \in G(F)\), \(u_F(g) = (Sn(g), a)\) is the class of quaternion algebra, where \(Sn\) is the spinor norm homomorphism \(G(F) \to F^\times/F^\times 2\) (cf. [11]).

**4. Invariants of simply connected groups**

We show in this section that a simply connected group has no nontrivial invariants in \(H^d[\mu_n^\otimes^{-1}]\) for \(d \leq 3\).

**4.1. Chow groups with coefficients.** Let \(G\) be a simply connected group defined over a field \(F\). Denote \(D(G)\) the free abelian group generated by the connected component of the Dynkin diagram of \(G\). In particular, \(D(G) = \mathbb{Z}\) if \(G\) is absolutely simple. There is a natural Galois action on \(D(G_{\text{sep}})\).

Let \(G\) be a split simply connected group over a field \(F\), \(T\) a split maximal torus in \(G\), \(W\) the Weyl group of \(G\). Let \(G = G_1 \times \cdots \times G_k\) be a product of simple subgroups. Then \(T^* = T_1^* \oplus \cdots \oplus T_k^*\) where \(T_i\) is a maximal torus in \(G_i\). There is a positive definite \(W\)-invariant quadratic form on \(T^*\), unique up to a multiple on each direct summand \(T_i^*\). Hence the group \(S^2(T^*)^W\) of \(W\)-invariant quadratic forms on \(T^*\) is canonically isomorphic to \(D(G)\).

**Proposition 4.1.** Let \(G\) be a split simply connected group over a field \(F\), and \(M\) a cycle module over \(F\). Then
\[
(i) \; H^0(G, M_j) = M_j(F);
(ii) \; H^1(G, M_j) = D(G) \otimes M_{j-2}(F).
\]
The first part and the case $j \leq 2$ of $(ii)$ is proved in [8]. Let $d = \dim G$ and $X = G/T$. Consider a spectral sequence (cf. [8])
\[ E^1_{p,q} = \Lambda^p T^* \otimes CH^{d-p-q} X \otimes M_{j+q-d}(F) \Rightarrow A^{d-p-q}(G, M_j). \]
It suffices to show that $E^2_{k,d-k-1}$ is trivial for all $k \neq 1$ and equals $D(G) \otimes M_{j-2}(F)$ if $k = 1$. This statement is proved for $k = 0, 1$ in [8] (with $D(G)$ replaced by $S^2(T^*W)$), so we can assume that $k \geq 2$.

The group $E^2_{k,d-k-1}$ is the homology of the complex
\[ E^1_{k+1,d-k-1} \rightarrow E^1_{k,d-k-1} \rightarrow E^1_{k-1,d-k-1}. \]
Consider Koszul complex
\[ C_i = \Lambda^i T^* \otimes S^{k+1-i} T^*. \]
The first Chern class defines an isomorphism $T^* \xrightarrow{\sim} CH^1(X)$ and therefore a homomorphism $S^l(T^*) \rightarrow CH^l(X)$ for any $l$. Thus, we have a commutative diagram
\[
\begin{array}{ccc}
C_{k+1} \otimes P & \longrightarrow & C_k \otimes P \\
\downarrow \alpha_{k+1} & & \downarrow \alpha_k \\
E^1_{k+1,d-k-1} & \longrightarrow & E^1_{k,d-k-1} \\
\end{array}
\]
where $P = M_{j-k-1}(F)$ with $\alpha_k$ and $\alpha_{k+1}$ being isomorphisms. Since the top row in the diagram is exact, in order to prove that the bottom row is also exact, it suffices to show that the restriction of $d$ on the kernel of $\alpha_{k-1}$ is injective.

Denote $H$ the kernel of the natural (split) surjection $S^2T^* \rightarrow CH^2(X)$. Clearly, $\ker \alpha_{k-1} = \Lambda^{k-1} T^* \otimes H \otimes P$. Hence it amounts to show that the composite

\[ (6) \]
\[ \Lambda^{k-1} T^* \otimes H \hookrightarrow \Lambda^{k-1} T^* \otimes S^2 T^* \xrightarrow{d'} \Lambda^{k-2} T^* \otimes S^3 T^* \]
is a split injection. The map $d'$ factors as the following composite

\[ \Lambda^{k-1} T^* \otimes S^2 T^* \xrightarrow{f} \Lambda^{k-2} T^* \otimes T^* \otimes S^2 T^* \xrightarrow{g} \Lambda^{k-2} T^* \otimes S^3 T^*, \]
where $f$ is the identity on $S^2 T^*$ and $g$ is the identity on $\Lambda^{k-2} T^*$. Hence the composite (6) factors as follows:

\[ \Lambda^{k-1} T^* \otimes H \longrightarrow \Lambda^{k-2} T^* \otimes T^* \otimes H \hookrightarrow \Lambda^{k-2} T^* \otimes T^* \otimes S^2 T^* \xrightarrow{g} \Lambda^{k-2} T^* \otimes S^3 T^*. \]
The first homomorphism is a split injection, hence it suffices to show that the restriction of $g$ on $\Lambda^{k-2} T^* \otimes T^* \otimes H$ is also a split injection. But $g$ is the identity on $\Lambda^{k-2} T^*$, hence we need to show that the restriction $h$ of $T^* \otimes S^2 T^* \rightarrow S^3 T^*$ on $T^* \otimes H$ is a split injection.

By [8, 4.8], the group $H$ coincides with $(S^2 T^*)^W$. Clearly, $H = H_1 \oplus \cdots \oplus H_k$ where $H_i = (S^2 T^*_i)^W_i$ for the Weyl group $W_i$ of $G_i$. The problem reduces to showing that the restriction of $T^*_i \otimes S^2 T^*_i \rightarrow S^3 T^*_i$ on $T^*_i \otimes H_i$ is a split injection, so we can assume that $G$ is a simple group. In this case $H$ is a cyclic group, generated by an integral quadratic form $q \in S^2 T^*$. Then $h$ is the multiplication
by $q$ and it is an injection since the symmetric algebra $S^*T^*$ is a domain and $h$ splits since $q$ is not zero modulo any prime number and therefore the cokernel of the restriction of $h$ has no torsion.

Now assume that $G$ is a simply connected (not necessarily split) group over $F$. By [8, Cor. B.3],
\[ A^1(G, K_2) \cong H^0(F, A^1(G_{\text{sep}}, K_2)). \]

Proposition 4.1 then gives

**Corollary 4.2.** $A^1(G, K_2) \cong H^0(F, D(G_{\text{sep}}))$.

By [12], $K_0(G) = \mathbb{Z}$, hence the first term $K_0(G)^{(1)}$ of the topological filtration is trivial. The natural homomorphisms $CH^i(G) \rightarrow K_0(G)^{(i/i+1)}$ are split by the Chern classes up to multiplication by $(i-1)!$. Hence, $(i-1)! \cdot CH^i(G) = 0$. In particular, $\text{Pic}(G) = CH^1(G) = 0 = CH^2(G)$ and $2 \cdot CH^3(G) = 0$.

The only possibly nontrivial differential in the Brown-Gersten-Quillen spectral sequence (cf. [21, Prop. 5.8])
\[ E_2^{p,q} = A^p(G, K_{-q}) \Rightarrow K_{-p-q}(G) \]
arriving to $CH^3(G)$ is a surjective homomorphism
\[ A^1(G, K_2) \twoheadrightarrow CH^3(G), \]
hence $CH^3(G)$ is a cyclic group of at most two elements, if $G$ is absolutely simple, since in this case $D(G_{\text{sep}}) = \mathbb{Z}$ and $A^1(G, K_2) = \mathbb{Z}$.

**Proposition 4.3.** Let $A$ be a central simple algebra over $F$, $G = SL_1(A)$. Then
\[ CH^3(G) = \begin{cases} 0 & \text{if ind}(A) \text{ is odd,} \\ \mathbb{Z}/2\mathbb{Z} & \text{if ind}(A) \text{ is even.} \end{cases} \]

**Proof.** Assume first that index of $A$ is odd. Choose a splitting field extension $L/F$ of $A$ of odd degree. In the split case the group $CH^3(G_L)$ is trivial by [27, Th. 2.7]. Since $CH^3(G)$ is a 2-torsion group, the natural homomorphism $CH^3(G) \rightarrow CH^3(G_L)$ is injective, hence $CH^3(G) = 0$.

Now let ind($A$) be even number. Assume first that $A$ is a quaternion (and hence division) algebra. The variety of $G$ is an affine quadric. Let $X$ be a projective quadric containing $G$ as an open subset, $Z = X \setminus G$. We have an exact sequence
\[ CH^2(Z) \rightarrow CH^3(X) \rightarrow CH^3(G) \rightarrow 0. \]
Since $X$ has a rational point, $CH^3(X) = \mathbb{Z}$. By assumption, the quadric $Z$ has no rational point, hence $CH^2(Z) = 2\mathbb{Z}$ (cf. [10] and [28]) and therefore $CH^3(G) = \mathbb{Z}/2\mathbb{Z}$.

Assume now that ind($A$) = 2, i.e. $A = M_n(Q)$, where $Q$ is quaternion division algebra. Consider $H = SL_1(Q)$ as a subgroup in $G = SL_n(Q)$ and the variety $X = G/H = GL_n(Q)/GL_1(Q)$. By Hilbert Theorem 90, $X(L) = GL_n(Q_L)/GL_1(Q_L)$ for any field extension $L/F$, hence the natural
map $G(L) \to X(L)$ is surjective and therefore the fiber of $\pi : G \to X$ over any geometric point is isomorphic to $\text{SL}_1(Q)$. In particular, the generic fiber $Y$ of $\pi$ is isomorphic to $H_E$, where $E = F(X)$. Since $X$ is a smooth variety having a rational point, we have $\text{ind}(Q_E) = \text{ind}(Q) = 2$. The surjectivity of the natural homomorphism

$$\text{CH}^3(G) \to \text{CH}^3(Y) \simeq \text{CH}^3(H_E) = \mathbb{Z}/2\mathbb{Z}$$

shows that $\text{CH}^3(G) \neq 0$, hence $\text{CH}^3(G) = \mathbb{Z}/2\mathbb{Z}$.

Consider the general case. Find a field extension $L/F$ such that $\text{ind}(A_L) = 2$. For example, $L$ can be taken as the function field of the generalized Severi-Brauer variety $SB(2, A)$ (cf. [4]). In the commutative diagram

$$
\begin{array}{ccc}
H^1(G, K_2) & \longrightarrow & \text{CH}^3(G) \\
\downarrow i & & \downarrow j \\
H^1(G_L, K_2) & \longrightarrow & \text{CH}^3(G_L)
\end{array}
$$

the horizontal homomorphisms are surjective and $i$ is an isomorphism, hence $j$ is surjective. By the case considered above, $\text{CH}^3(G_L) = \mathbb{Z}/2\mathbb{Z}$, hence $\text{CH}^3(G)$ is not trivial, and therefore $\text{CH}^3(G) = \mathbb{Z}/2\mathbb{Z}$. \hfill \Box

**Remark 4.4.** The proof shows that if $L/F$ is a field extension such that the algebra $A_L$ is of even index, then the natural homomorphism $\text{CH}^3(G) \to \text{CH}^3(G_L)$ is an isomorphism of groups of order 2.

**Lemma 4.5.** Let $n$ be prime to $\text{char } F$. For any $i \geq 0$ there exists an exact sequence

$$A^i(G, K_{i+1}) \xrightarrow{n} A^i(G, K_{i+1}) \to A^i(G, H^{i+1}(-n^i\mu)) \to n \text{CH}^{i+1}(G) \to 0.$$

In particular,

$$A^i(G, H^{i+1}(-n^i\mu)) = A^i(G, K_{i+1})/nA^i(G, K_{i+1})$$

for $i = 0$ and 1.

**Proof.** Follows by diagram chase in

$$
\begin{array}{ccc}
\prod_{X_{i-1}} K_2 F(x) & \longrightarrow & \prod_{X_i} K_1 F(x) \\
\downarrow n & & \downarrow n \\
\prod_{X_{i-1}} K_2 F(x) & \longrightarrow & \prod_{X_i} K_1 F(x) \\
\downarrow h_2 & & \downarrow h_1 \\
\prod_{X_{i-1}} H^2 F(x) & \longrightarrow & \prod_{X_i} H^1 F(x) \\
\downarrow h_2 & & \downarrow h_1 \\
\prod_{X_{i-1}} H^2 F(x) & \longrightarrow & \prod_{X_i} H^1 F(x)
\end{array}
$$

where the cohomology groups are taken with coefficients $\mu^2_n$, $\mu_n$ and $\mathbb{Z}/n\mathbb{Z}$ respectively, and surjectivity of $h_2$ (cf. [17]). \hfill \Box
4.2. Étale cohomology. Let $G$ be a simply connected group defined over a field $F$ and let $N$ be a Galois module of bounded exponent. We have the following computation of étale cohomology groups over $F_{\text{sep}}$.

**Lemma 4.6.** There are following isomorphisms of Galois modules:

$$H^m_{\text{et}}(G_{\text{sep}}, N) \simeq \begin{cases} N & \text{if } m = 0, \\ 0 & \text{if } m = 1, 2, 4, \\ D(G_{\text{sep}}) \otimes N[-2] & \text{if } m = 3. \end{cases}$$

**Proof.** The case $m = 0$ is trivial. Since $G_{\text{sep}}^* = 0$ and $\text{Pic}(G_{\text{sep}}) = 0$, it follows from (4), that $H^1_{\text{et}}(G_{\text{sep}}, \mu_n) = 0$ for any $n$ prime to $\text{char } F$ and hence $H^1_{\text{et}}(G_{\text{sep}}, N) = 0$ for any $N$ of bounded exponent.

The Kummer sequence induces an exact sequence

$$0 \rightarrow \text{Pic}(G_{\text{sep}})/n \rightarrow H^2_{\text{et}}(G_{\text{sep}}, \mu_n) \rightarrow n \text{ Br}(G_{\text{sep}}).$$

Since $n \text{ Br}(G_{\text{sep}}) = 0$ by [9, Th. 4.1], $H^2_{\text{et}}(G_{\text{sep}}, \mu_n) = 0$ for any $n$ prime to $\text{char } F$ and hence $H^2_{\text{et}}(G_{\text{sep}}, N) = 0$ for any $N$ of bounded exponent.

The Bloch-Ogus spectral sequence (2) for $N$ gives an exact sequence

$$0 \rightarrow A^1(G_{\text{sep}}, \mathcal{H}^2(N)) \rightarrow H^3_{\text{et}}(G_{\text{sep}}, N) \rightarrow A^0(G_{\text{sep}}, \mathcal{H}^3(N)).$$

By Proposition 4.1, the last group in this sequence is trivial and the first is isomorphic to $D(G_{\text{sep}}) \otimes N[-2]$.

In order to show that $H^1_{\text{et}}(G_{\text{sep}}, N)$ is trivial, it suffices to prove that the groups $A^2(G_{\text{sep}}, \mathcal{H}^2(N))$, $A^1(G_{\text{sep}}, \mathcal{H}^3(N))$ and $A^0(G_{\text{sep}}, \mathcal{H}^4(N))$ staying on the corresponding diagonal in the Bloch-Ogus spectral sequence (2), are trivial. The first group is isomorphic to $CH^2(G_{\text{sep}}) \otimes N[-2]$ and hence trivial. The last two groups are trivial by Proposition 4.1. □

Since the natural homomorphisms $H^p(F, N) \rightarrow H^p_{\text{et}}(G, N)$ are injective, all the differentials in the Hochschild-Serre spectral sequence

$$E^2_{p,q} = H^p(F, H^q_{\text{et}}(G_{\text{sep}}, N)) \Rightarrow H^{p+q}_{\text{et}}(G, N),$$

arriving to $E^{p,0}_2$, are trivial. By Lemma 4.6, $E^{p,q}_2 = 0$ if $q = 1$ or 2 and $E^{p,3}_2 = H^p(F, D(G_{\text{sep}}) \otimes N[-2])$.

Thus, we have proved

**Proposition 4.7.** Let $G$ be a simply connected group defined over a field $F$ and let $N$ be a Galois module of bounded exponent. Then

$$\tilde{H}^m_{\text{et}}(G, N) = \begin{cases} 0 & \text{if } m = 0, 1, 2, \\ H^0(F, D(G_{\text{sep}}) \otimes N[-2]) & \text{if } m = 3 \\ H^1(F, D(G_{\text{sep}}) \otimes N[-2]) & \text{if } m = 4. \end{cases}$$

Let $n \in \mathbb{N}$ be prime to $\text{char } F$. The Bloch-Ogus spectral sequence for the Galois module $\mu_n^{\otimes 2}$ gives the following exact sequence

$$0 \rightarrow A^1(G, \mathcal{H}^2(\mu_n^{\otimes 2})) \xrightarrow{e_2} \tilde{H}^3_{\text{et}}(G, \mu_n^{\otimes 2}) \rightarrow \tilde{A}^0(G, \mathcal{H}^3(\mu_n^{\otimes 2})) \xrightarrow{d_2} A^2(G, \mathcal{H}^2(\mu_n^{\otimes 2}))$$
Lemma 4.8. Let $G$ be a simply connected group. Then $e_2$ is an isomorphism and $A^0(G, \mathcal{H}^3(\mu_n^{\otimes 2})) = 0$.

Proof. In the commutative diagram

$$
\begin{array}{c}
A^1(G, K_2) \longrightarrow A^1(G, \mathcal{H}^2(\mu_n^{\otimes 2})) \longrightarrow \tilde{H}_c^2(G, \mu_n^{\otimes 2}) \\
\downarrow 1 \quad \downarrow 1 \\
H^0(F, D(G_{\text{sep}})) \longrightarrow H^0(F, D(G_{\text{sep}})/n)
\end{array}
$$

the vertical isomorphisms are given by Corollary 4.2 and by Proposition 4.7. The group $H^0(F, D(G_{\text{sep}})/n)$ is isomorphic to $H^0(F, D(G_{\text{sep}}))/n$ since the Galois module $D(G_{\text{sep}})$ is permutation. Hence the bottom homomorphism and therefore $e_2$ are surjective. Thus, $d_2$ is injective and $A^0(G, \mathcal{H}^3(\mu_n^{\otimes 2})) = 0$ since $A^2(G, \mathcal{H}^2(\mu_n^{\otimes 2})) = \text{CH}^2(G)/n = 0$.

An application of Lemma 4.8 is the following

Proposition 4.9. For a simply connected group $G$, $\text{Inv}^d(G, H^*[N]) = 0$ for any Galois module $N$ of bounded exponent, $d = 0, 1$, and for $N = \mu_n^{\otimes -1}$ ($n$ prime to char $F$), $d \leq 3$.

Proof. The case $d = 0$ has been considered in Corollary 3.2. The cases $d = 1$ and $d = 2$ follow from Corollaries 3.5 and 3.14. Finally, if $d = 3$, by Lemmas 1.9, 4.8 and Theorem 2.3, $\text{Inv}^3(G, H^*[\mu_n^{\otimes -1}]) = 0$. \hfill \Box

5. Cohomological invariants of $\text{SL}_1(A)$

It is proved in section 4 that simply connected groups have no nonzero invariants in $H^*[\mu_n^{\otimes -1}]$ of dimension at most 3 and split simply connected groups have no nontrivial invariants at all. We show that the group $\text{SL}_1(A)$ still has no invariant if index of $A$ is 2. For a biquaternion algebra $A$ of index 4 we prove that there is the only nontrivial invariant in $H^4[\mathbb{Z}/2\mathbb{Z}]$, namely the Rost’s invariant.

5.1. Invariants of $\text{SL}_1(A)$ with $\text{ind}(A) = 2$.

Lemma 5.1. Let $G = \text{SL}_1(Q)$, where $Q$ is a quaternion algebra over $F$. Then $A^0(G, M) = M(F)$ for any cycle module $M$ over $F$.

Proof. Let $X$ and $Z$ be projective quadrics as in the proof of Proposition 4.3 and let $C$ be projective conic curve, corresponding to $Q$. Then $Z \simeq C \times C$ and $X$ (being a projective quadric with a rational point) contains a hyperplane section $Y$, such that $X \setminus Y$ is isomorphic to an affine space $A^3_F$. We have the localization exact sequence (cf. [24, Sec. 5])

$$0 \longrightarrow A^0(X, M_d) \longrightarrow A^0(G, M_d) \longrightarrow A^0(Z, M_{d-1}) \longrightarrow A^1(X, M_d).$$

Another localization sequence for the pair $(X, Y)$ immediately gives

$$A^0(X, M_d) = A^0(A^3_F, M_d) = M_d(F) \quad \text{and} \quad A^1(X, M_d) \simeq A^1(Y, M_{d-1}).$$
A spectral sequence ([24, Cor. 8.2]) associated to a projection \( Z \to C \) identifies \( A^0(Z, M_{d-1}) \) with \( A^0(C, M_{d-1}) \).

On the other hand, \( Y - pt \) is a vector bundle over \( C \). Hence, by [24, Prop. 8.6],

\[
A^1(X, M_d) \cong A^0(Y, M_{d-1}) = A^0(Y - pt, M_{d-1}) \cong A^0(C, M_{d-1}).
\]

Therefore, it suffices to show commutativity of the diagram

\[
\begin{array}{ccc}
A^0(Z, M_{d-1}) & \longrightarrow & A^1(X, M_d) \\
\downarrow & & \downarrow \\
A^0(C, M_{d-1}) & \longrightarrow & A^0(C, M_{d-1}).
\end{array}
\]

Since all homomorphisms in the diagram are natural (given by four basic maps in [24]), it is sufficient to prove the statement in the case \( d = 1 \) and \( M \) is given by Milnor \( K \)-groups. In other words, we have to check that the classes of \( Z \) and \( Y \) coincide in \( A^1(X, K_1) = CH^1(X) \). But these subvarieties in \( X \) are hyperplane sections, hence are rationally equivalent. \( \square \)

**Theorem 5.2.** Let \( G = SL_1(A) \), where \( A \) is a central simple algebra of index at most 2. Then \( \text{Inv}^d(G, M) = 0 \) for any \( d \) and a cycle module \( M \).

**Proof.** The case of a split algebra \( A \) was considered in Corollary 2.6. Hence we may assume that \( G = SL_n(Q) \), where \( Q \) is a quaternion division algebra. The case \( n = 1 \) follows from Theorem 2.3 and Lemma 5.1. In the general case the group of \( F \)-points of \( G \) is generated by \( SL_1(Q) \), embedded to \( G \) by \( a \mapsto \text{diag}(a, 1, \ldots, 1) \) and unipotent subgroups of elementary matrices. By Proposition 2.4, the restriction of any invariant of \( G \) on all there subgroups is trivial, hence so is the invariant. \( \square \)

### 5.2. Invariants of dimension 4

Let \( A \) be a biquaternion algebra (tensor product of two quaternion algebras) over a field \( F \) of characteristic different from 2. There is a 6-dimensional quadratic form \( q \) associated to \( A \) (unique up to a scalar multiple), called an *Albert form* (cf. [11, 16.A]). This form defines a 4-dimensional projective quadric hypersurface, which we call the *Albert quadric* of \( A \). The Albert quadric is the only 4-dimensional projective homogeneous variety of the spinor group \( \text{Spin}(q) \) (cf. [19]). This simply connected absolutely simple group of type \( D_3 \) is naturally isomorphic to \( GL_1(A) \), the one of type \( A_3 \) (cf. [11, 26.B]). The only 4-dimensional projective homogeneous variety of \( GL_1(A) \) is the generalized Severi-Brauer variety \( SB(2, A) \) of right 8-dimensional ideals in \( A \), so that the Albert quadric of \( A \) is isomorphic to \( SB(2, A) \).

M. Rost (cf. [13]) has constructed an invariant of \( G = SL_1(A) \) in \( H^4[Z/2Z] \). He also computed the kernel and the image of the invariant. More precisely, the following holds (with \( SK_1(A) = SL_1(A)/[A^\times, A^\times] \)).
Theorem 5.3. (Rost) Let $X$ be an Albert quadric of $A$. Then there is an exact sequence

$$0 \longrightarrow \SK_1(A) \longrightarrow H^4(F, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^4(F(X), \mathbb{Z}/2\mathbb{Z}).$$

An elementary construction of Rost’s invariant is given in [11, §17].

If $A$ is not a division algebra, then Rost’s invariant is trivial by Theorem 5.2. If $A$ is division, then it is shown in [14] that the group $\SK_1(A_F(G))$ is not trivial, hence the Rost’s invariant is not trivial.

Our aim is to prove that Rost’s invariant is the only nontrivial degree 4 invariant of $G$. More precisely, we prove

Theorem 5.4. Let $G = \text{SL}_1(A)$, where $A$ is a biquaternion division algebra. Then $\text{Inv}^4(G, H^4[\mathbb{Z}/2\mathbb{Z}]) = \mathbb{Z}/2\mathbb{Z}$ is generated by the Rost’s invariant.

Since $A^2(G, \mathbb{H}^2) = \text{CH}^2(G)/2 = 0$ (cohomology groups are taken with coefficients $\mathbb{Z}/2\mathbb{Z}$), the Bloch-Ogus spectral sequence for $\mathbb{Z}/2\mathbb{Z}$ gives the following exact sequence

$$0 \longrightarrow A^1(G, \mathbb{H}^3) \xrightarrow{e_3} \widetilde{H}_A^4(G) \longrightarrow \widetilde{A}^0(G, \mathbb{H}^4) \xrightarrow{d_3} A^2(G, \mathbb{H}^3).$$

Lemma 5.5. $e_3$ is an isomorphism. In particular, $d_3 : \widetilde{A}^0(G, \mathbb{H}^4) \to A^2(G, \mathbb{H}^3)$ is injective.

Proof. In the commutative diagram

$$
\begin{array}{ccc}
A^1(G, \mathbb{H}^2) \otimes \mathbb{F}^\times & \xrightarrow{e_2 \otimes \text{id}} & \widehat{H}_A^3(G) \otimes \mathbb{F}^\times \\
\downarrow m & & \downarrow \\
A^1(G, \mathbb{H}^3) & \xrightarrow{e_3} & \widehat{H}_A^4(G) \\
\end{array}
$$

the vertical homomorphisms are the natural product maps and the right horizontal isomorphisms are given by Proposition 4.7. Since $G$ is absolutely simple, $D(G_{\text{sep}}) = \mathbb{Z}$ and hence the right vertical homomorphism is an isomorphism. By Lemma 4.8, $e_2$ is an isomorphism, hence $e_3$ is surjective and therefore is an isomorphism. Note that $m$ is also an isomorphism.

Corollary 5.6. The natural homomorphism $A^1(G, K_3) \to A^1(G, \mathbb{H}^3)$ is surjective.

Proof. In the commutative diagram

$$
\begin{array}{ccc}
A^1(G, K_3) \otimes \mathbb{F}^\times & \longrightarrow & A^1(G, \mathbb{H}^2) \otimes \mathbb{F}^\times \\
\downarrow & & \downarrow m \\
A^1(G, K_3) & \longrightarrow & A^1(G, \mathbb{H}^3) \\
\end{array}
$$

the upper horizontal homomorphism is surjective by Lemma 4.5 and $m$ is an isomorphism (cf. the proof of Lemma 5.5), hence the result.

Lemma 5.7. $A^2(G, K_3)$ is an infinite cyclic group.
Proof. By [27, Th. 2.7], $A^2(G, K_3) = \mathbb{Z}$ in the split case. Hence, it suffices to show that in the general case $A^2(G, K_3)$ has no torsion. Since there is a splitting field of $A$ of degree 4, it is sufficient to prove that $2A^2(G, K_3) = 0$.

Consider the following diagram with exact columns (cf. [17])
\[
\begin{array}{ccccccc}
\mu_2 \otimes \prod_{G^{(1)}} K_1 F(g) & \xrightarrow{d} & \mu_2 \otimes \prod_{G^{(2)}} K_0 F(g) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\prod_{G^{(1)}} K_2 F(g) & \rightarrow & \prod_{G^{(2)}} K_1 F(g) & \rightarrow & \prod_{G^{(3)}} K_0 F(g) \\
\downarrow & & \downarrow 2 & & \downarrow 2 & & \downarrow \\
K_3 F(G) & \rightarrow & \prod_{G^{(1)}} K_2 F(g) & \rightarrow & \prod_{G^{(2)}} K_1 F(g) & \rightarrow & \prod_{G^{(3)}} K_0 F(g) \\
\downarrow h_3 & & \downarrow h_2 & & \downarrow h_1 \\
H^3 F(G) & \rightarrow & \prod_{G^{(1)}} H^2 F(g) & \rightarrow & \prod_{G^{(2)}} H^1 F(g). \\
\end{array}
\]

The homomorphism $d$ is surjective since $\text{CH}^3(G) = 0$. The surjectivity of $h_3$ is proved in [18] and [23]. Now the result follows from Corollary 5.6 by diagram chase.

Proof of Theorem 5.4. By Lemma 5.5,
\[d_3: \tilde{A}^0(G, \mathcal{H}^4) \rightarrow A^2(G, \mathcal{H}^3)\]
is injective. Denote $j$ the injective composite
\[\text{Inv}^4(G, H^*[\mathbb{Z}/2\mathbb{Z}]) \hookrightarrow \tilde{A}^0(G, \mathcal{H}^4) \xrightarrow{d_3} A^2(G, \mathcal{H}^3).\]
Let $L/F$ be a field extension such that $\text{ind}(A_L) = 2$. Consider the following commutative diagram
\[
\begin{array}{ccccccc}
\text{Inv}^4(G, H^*[\mathbb{Z}/2\mathbb{Z}]) & \xrightarrow{j} & A^2(G, \mathcal{H}^3) & \xrightarrow{k} & 2 \text{CH}^3(G) \\
\downarrow & & \downarrow i & & \downarrow l \\
\text{Inv}^4(G_L, H^*[\mathbb{Z}/2\mathbb{Z}]) & \xrightarrow{j_L} & A^2(G_L, \mathcal{H}^3) & \xrightarrow{k_L} & 2 \text{CH}^3(G_L). \\
\end{array}
\]
where $k$ is defined in Lemma 4.5. The group $\text{Inv}^4(G_L, H^*[\mathbb{Z}/2\mathbb{Z}])$ is trivial by Theorem 5.2, hence $i \circ j = 0$. Lemma 4.5 shows that $k$ is surjective and $l$ is an isomorphism of groups of order 2 by Remark 4.4. Therefore, $i \neq 0$ and $j$ is not surjective.

By Lemma 4.5, the sequence
\[0 \rightarrow A^2(G, K_3)/2A^2(G, K_3) \rightarrow A^2(G, \mathcal{H}^3) \xrightarrow{k} 2 \text{CH}^3(G) \rightarrow 0\]
is exact. Proposition 4.3 and Lemma 5.7 show that $A^2(G, \mathcal{H}^3)$ is a group of order 4. Since $j$ is not surjective, the group $\text{Inv}^4(G, H^*[\mathbb{Z}/2\mathbb{Z}])$ is of at most 2 elements. On the other hand, the Rost invariant is not trivial, hence $\text{Inv}^4(G, H^*[\mathbb{Z}/2\mathbb{Z}]) = \mathbb{Z}/2\mathbb{Z}$. \qed
Remark 5.8. It follows from Lemma 5.5 that the homomorphism $\tilde{H}_1^*(G) \to \tilde{A}^0(G, \mathcal{H})$ is trivial. This implies that the Rost’s invariant is not “global”, i.e. cannot be extended to a natural transformation of functors $G$ and $H^4(\ast, \mathbb{Z}/2\mathbb{Z})$ from the category $\mathcal{F}$-$\text{alg}$ to $\mathcal{G}$-$\text{roups}$. That is why we define an invariant as a natural transformation of functors defined on a smaller category $\mathcal{F}$-$\text{fields}$.

6. Degree Four Algebras

The aim of this section is to generalize Theorem 5.3 to the case of arbitrary central simple algebra of dimension 16. We assume that $\text{char} F \neq 2$ for the base field $F$.

Let $A$ and $B$ be two central simple algebras over $F$. If they are similar (determine the same element in the Brauer group $\text{Br}(F)$), then the groups $K_1(A)$ and $K_1(B)$ are canonically isomorphic.

Any anti-automorphism $\varphi$ of $A$ induces the identity automorphism on $K_1(A)$ since for any $a \in A^\times$ the elements $a$ and $\varphi(a)$ have the same minimal polynomials and therefore are conjugate. Hence, if $A$ and $B$ are anti-similar (i.e. $A$ is similar to $B^{op}$), then $K_1(A)$ and $K_1(B)$ are also canonically isomorphic.

Let $A$ be a central simple algebra of degree 4. The algebra $A \otimes_F A$ is then similar to a quaternion algebra $Q$. Denote $C$ the corresponding conic curve. Since $A$ is of exponent dividing 4, the algebra $A \otimes_F Q$ is similar to $A^{op}$ and hence is anti-similar to $A$. Denote $i$ the canonical isomorphism

$$K_1(A \otimes_F Q) \xrightarrow{i} K_1(A).$$

For a variety $X$ over $F$ we write $K_4(X, A)$ for the $K$-groups of the category of locally free right $\mathcal{O}_X \otimes_F A$-modules of finite rank.

By a computation of $K$-theory of Severi-Brauer varieties (cf. [21], [20]),

$$K_1(C, A) \xrightarrow{\sim} K_1(A) \oplus K_1(A \otimes Q) \xrightarrow{\sim} K_1(A) \oplus K_1(A).$$

(7)

Consider the case when $Q$ splits, i.e. $C$ is a projective line. The inverse isomorphism to (7) is given by the inverse image with respect to the structure morphism $p : C \to \text{Spec}(F)$ on the first component and on the second component by $u \mapsto p^* (u \otimes \mathcal{O}(-1))$. Since the class of a rational point in $K_0(C)$ equals $[\mathcal{O}] - [\mathcal{O}(-1)]$, for any rational point $x \in C$ the direct image homomorphism

$$K_1(A) = K_1(\text{Spec} F(x), A) \to K_1(C, A) \xrightarrow{i} K_1(A) \oplus K_1(A)$$

takes $a$ to $([a], -[a])$.

In the general case (when $Q$ is not necessarily split) consider any closed point $x \in C$. The direct image homomorphism

$$K_1(A_F(x)) \xrightarrow{i_{x*}} K_1(C, A) \xrightarrow{\sim} K_1(A) \oplus K_1(A)$$

factors as follows:

$$K_1(A_F(x)) \to K_1(C_F(x), A_F(x)) \xrightarrow{N_{F(x)/F}} K_1(C, A) \xrightarrow{\sim} K_1(A) \oplus K_1(A).$$

Hence the computation above proves the following
Lemma 6.1. For any closed point \( x \in C \), the sum of \( K_1(A) \)-coordinates in the image of \( i_x \) is zero.

Consider the inverse image homomorphism with respect to the generic point \( q : \text{Spec} \, F(C) \to C \) of \( C \):

\[
K_1(A) \oplus K_1(A \otimes Q) = K_1(C, A) \xrightarrow{q^*} K_1(A_{F(C)}).
\]

The restriction of \( q^* \) on the first component \( K_1(A) \) is the field extension homomorphism. The restriction of \( q^* \) on the second component \( K_1(A \otimes Q) \) is given by the following composite of natural homomorphisms (we use the fact that \( Q \) splits over \( F(C) \)):

\[
K_1(A \otimes Q) \to K_1(A_{F(C)} \otimes Q_{F(C)}) \to K_1(A_{F(C)}).
\]

Hence the composite of the second restriction with the natural identification \( K_1(A \otimes Q) = K_1(A) \) is also the field extension homomorphism. Thus, we have proved

Lemma 6.2. The image of \( q^* \) coincides with the image of the natural field extension homomorphism \( K_1(A) \to K_1(A_{F(C)}) \).

Now consider the localization sequence (cf. [21]):

\[
\prod_{x \in C} K_1(A_{F(x)}) \xrightarrow{i_x^*} K_1(C, A) \xrightarrow{q^*} K_1(A_{F(C)}) \to \prod_{x \in C} K_0(A_{F(x)}).
\]

Proposition 6.3. The natural homomorphism \( SK_1(A) \to SK_1(A_{F(C)}) \) is an isomorphism.

Proof. By 6.1 and 6.2, the rows of the following diagram with reduced norm homomorphisms

\[
\begin{array}{c}
0 \to K_1(A) \to K_1(A_{F(C)}) \to \prod_{x \in C} K_0(A_{F(x)}) \\
\downarrow \text{Nrd} \quad \quad \quad \downarrow \text{Nrd} \quad \quad \quad \downarrow \text{Nrd} \\
0 \to K_1(F) \to K_1(F(C)) \to \prod_{x \in C} K_0(F(x)).
\end{array}
\]

are exact. The result follows from the injectivity of the right vertical homomorphism and the snake lemma.

The Bloch-Ogus spectral sequence for \( C \) becomes a long exact sequence of cohomology groups with coefficients \( \mathbb{Z}/2\mathbb{Z} \):

\[
\cdots \to \prod_{x \in C} H^{n-2}F(x) \to H^{n-1}_{et}C \to H^nF(C) \xrightarrow{d} \prod_{x \in C} H^{n-1}F(x) \to \cdots
\]

Since \( Q \) splits over \( F(C) \), the class of \( A_{F(C)} \) belongs to \( H^2F(C) \). It follows from \( d(A_{F(C)}) = 0 \) that there is \( \theta \in H^2_{et}C \) such that \( \theta_{F(C)} = A_{F(C)} \).

Assume that \( Q \) is a division algebra. Then \( \theta \) is not in the image of \( H^2F \). By [26], there is an exact sequence

\[
\cdots \to H^nF \to H^n_{et}C \to H^{n-2}F \xrightarrow{\partial} H^{n+1}F \to \cdots,
\]
where $\partial$ is the multiplication by $(-1) \cup [Q] \in H^3F$. In our case the class of $Q$ is divisible by 2 in the Brauer group, hence $(-1) \cup [Q] = 0$. Thus, for any $n$ there is an exact sequence

$$0 \to H^nF \to H^nC \to H^{n-2}F \to 0.$$ 

Since the class $\theta$ does not come from $H^2F$, its image in $H^0F$ is not trivial. Hence any element in $H^nC$ can be written in the form $v_C + u_C \cup \theta$ for $v \in H^nF$ and $u \in H^{n-2}F$.

It follows from exactness of (8) that the natural map

$$H^nC \to A^0(C, \mathcal{H}^n)$$

is surjective. We have proved

**Lemma 6.4.** Any element in $A^0(C, \mathcal{H}^n)$ is of the form $v_{F(C)} + u_{F(C)} \cup A_{F(C)}$ for $v \in H^nF$ and $u \in H^{n-2}F$. \qed

Let $X$ be the generalized Severi-Brauer variety $SB(2, A)$. For any point $x \in X$, the index of $A$ over $F(x)$ is at most 2. Hence the algebra $Q$ splits over $F(x)$ and therefore $C_{F(x)}$ is a projective line.

By [2, Ch.XI, Th.9], there is a quadratic subfield $L \subset A$. Since $\text{ind}(A_L) = 2$, there is an $L$-rational point in $X_L$. The image of this point to $X$ gives a closed point of degree 2.

**Lemma 6.5.**

$$\text{Ker}(A^0(C, \mathcal{H}^4) \to A^0(C_{F(X)}, \mathcal{H}^4)) \subset \text{Im}(H^4F \to H^4F(C)).$$

**Proof.** Let $w$ be in the kernel. By Lemma 6.4, $w = v_{F(C)} + u_{F(C)} \cup A_{F(C)}$ for some $v \in H^4F$ and $u \in H^2F$. Choose a closed point $x \in X$ of degree 2. The element $t = v_{F(x)} + u_{F(x)} \cup A_{F(x)} \in H^4F(x)$ is split by the extension $F(x)/(C \times X)$. Since $C \times X$ has a point over $F(x)$, the map $H^4F(x) \to H^4F(x)(C \times X)$ is injective and hence $t = 0 \in H^4F(x)$, i.e. $u_{F(x)} \cup A_{F(x)} = v_{F(x)}$.

Choose an element $u' \in H^2(F, \mu_4^{\otimes 2})$ in the inverse image of $u$ under the surjection $H^2(F, \mu_4^{\otimes 2}) \to H^2F$ (cf. [17]) and consider the cup-product $s = u' \cup [A]$ with respect to the pairing

$$H^2(F, \mu_4^{\otimes 2}) \otimes H^2(F, \mu_4) \to H^4(F, \mu_4^{\otimes 3}).$$

We know that $s_{F(x)} = v_{F(x)}$ and $v$ is an element of order 2. Hence

$$0 = 2v = N_{F(x)/F}(v_{F(x)}) = N_{F(x)/F}(s_{F(x)}) = 2s = u' \cup [Q] \in H^4(F, \mu_4^{\otimes 3}).$$

Since the natural homomorphism $H^4(F) \to H^4(F, \mu_4^{\otimes 3})$ is injective (cf. [18], [23]), we have $u \cup [Q] = 0 \in H^4F$. By [15, Prop.3.15], $u$ belongs to the image of the norm map $\prod_{x \in C} H^2F(x) \to H^2F$. The exactness of the sequences (8) and (9) shows then that $u_{F(C)} \cup A_{F(C)} \in \text{Im}(H^4F \to H^4F(C))$, hence $w \in \text{Im}(H^4F \to H^4F(C))$. \qed

**Theorem 6.6.** Let $A$ be a central simple algebra of dimension 16 over $F$, $X = SB(2, A)$. Then there exists an exact sequence

$$0 \to \text{SK}_1(A) \to H^4F/(2|A| \cup H^2F) \to H^4F(X).$$
Proof. By Theorem 5.3 such a sequence exists if $2[A] = 0 \in \text{Br}(F)$. In the general case, $A$ is a division algebra and $2[A_F(C)] = 0 \in \text{Br}(F(C))$, hence there is a sequence

$$0 \rightarrow \text{SK}_1(A_F(C)) \rightarrow H^4F(C) \rightarrow H^4F(C \times X).$$

Proposition 6.3 shows that the first term of this sequence if isomorphic to $\text{SK}_1(A)$. Hence it suffices to prove the exactness of the sequence

$$H^2F \xrightarrow{[2A]} \text{Ker}(H^4F \rightarrow H^4F(X)) \rightarrow \text{Ker}(H^4F(C) \rightarrow H^4F(C \times X)) \rightarrow 0.$$ 

By [15, Prop.3.15], the kernel of $H^4F \rightarrow H^4F(C)$ equals $2[A] \cup H^2F$, whence the exactness in the second term. Finally take any $w \in \text{Ker}(H^4F(C) \rightarrow H^4F(C \times X))$. For any closed point $x \in C$ consider the following commutative diagram

$$
\begin{array}{ccc}
H^4F(C) & \longrightarrow & H^4F(C \times X) \\
\downarrow \partial_x & & \downarrow \partial_y \\
H^3F(x) & \longrightarrow & H^3F(x)(X)
\end{array}
$$

where $\{x\} = x \times X$. Since $Q$ is split over $F(x)$, $X$ is an Albert quadric over $F(x)$. Anisotropic Albert form cannot be a subform of a 3-fold Pfister form, hence by a theorem of Arason [3], the bottom homomorphism is injective; therefore $\partial_x(w) = 0$ for any closed $x \in C$, i.e. $w \in A^0(C, H^4)$. By Lemma 6.5, $w = v_F(C)$ for some $v \in H^4F$. It remains to notice that, since $C$ has a point over $F(X)$, the natural homomorphism $H^4F(X) \rightarrow H^4F(C \times X)$ is injective and therefore $v \in \text{ker}(H^4F \rightarrow H^4F(X))$. 

\[ \blacksquare \]

References

Actions of algebraic groups. Then the closed orbits are the origin and the hyperbolae \( xy = c \), where \( c = 0 \). The other orbit closures are the coordinate axes. 3) The natural action of SL2 on \( \mathbb{C}^2 \) has 2 orbits: the origin and its complement. 2.1. Representations of connected reductive groups and U-invariants. Given an algebraic group \( G \) and two \( G \)-modules \( V, W \), we denote by \( \text{Hom}_G(V, W) \) the vector space of morphisms of \( G \)-modules (i.e., equivariant linear maps) \( f : V \rightarrow W \). Observe that \( \text{Hom}_G(V, \mathbb{C}[X]) \cong \mathbb{C}[X] \otimes V \stackrel{\sim}{\rightarrow} \text{Mor}_G(X, V^\ast) \). Let \( G \) be a reductive group acting on an algebraic variety \( X \) over \( \text{char } k = 0 \). And \( \pi : X \rightarrow \text{Spec} \mathbb{C}[X]^G \) is the categorical quotient. Prove that for \( Z_1, Z_2 \) invariant closed subsets such that \( Z_1 \cap Z_2 = \emptyset \) we have \( \pi(Z_1) \cap \pi(Z_2) = \emptyset \). 8 Let \( G \) be a reductive group acting on an algebraic variety \( X \) over \( \text{char } k = 0 \). Prove that the map \( \pi : X \rightarrow \text{Spec} \mathbb{C}[X]^G \) is surjective. Is it true when \( G \) is not reductive but the algebra \( k[X]^G \) is still finitely generated. 9 Let \( G \) be a reductive group acting on an algebraic variety \( X \) over \( \text{char } k = 0 \). Consider the categorical quotient \( \tilde{\pi} : X \rightarrow \text{Spec} \mathbb{C}[X]^G \). Is it proper? Equidimension every algebraic representation of semisimple algebraic group is isomorphic to the action of this group on the module of holomorphic sections of some reductive bundle over homogeneous space. Using this, we give a complete description of the field of differential invariants for this action. And obtain a criterion, which separates regular orbits. 2000 MSC: 22E26, 32L05, 53A55. Keywords: semisimple algebraic group, algebraic representation, reductive homogeneous bundle, invariant connection, differential invariant, jet space. Invariants for the separation of the orbits of algebraic groups actions. In the present paper we discuss a general problem of classification of the orbits for the action of an arbitrary semi-simple, complex algebraic group in its algebraic representation.